BASE SPACES OF NON-ISOTRIVIAL FAMILIES OF SMOOTH MINIMAL MODELS

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Für Hans Grauert, mit tiefer Bewunderung.

Given a polynomial h of degree n let $\mathcal{M}_h$ be the moduli functor of canonically polarized complex manifolds with Hilbert polynomial h. By [23] there exist a quasi-projective scheme M_h together with a natural transformation

$$\Psi : \mathcal{M}_h \to \mathrm{Hom}(\_, M_h)$$

such that M_h is a coarse moduli scheme for $\mathcal{M}_h$. For a complex quasi-projective manifold U we will say that a morphism $\varphi : U \to M_h$ factors through the moduli stack, or that φ is induced by a family, if φ lies in the image of $\Psi(U)$, hence if $\varphi = \Psi(f : V \to U)$.

Let Y be a projective non-singular compactification of U such that $S = Y \setminus U$ is a normal crossing divisor, and assume that the morphism $\varphi : U \to M_h$, induced by a family, is generically finite. For moduli of curves of genus $g \geq 2$, i.e. for $h(t) = (2t-1)(g-1)$, it is easy to show, that the existence of φ forces $\Omega^1_Y(\log S)$ to be big (see 1.1 for the definition), hence that $S^m(\Omega^1_Y(\log S))$ contains an ample subsheaf of full rank for some $m > 0$. In particular, U should not be an abelian variety or $\mathbb{C}^*$. By [15] the bigness of $\Omega^1_Y(\log S)$ implies even the Brody hyperbolicity of U. As we will see, there are other restrictions on U, as those formulated below in 0.1, 0.3, and 0.2.

In the higher dimensional case, i.e. if $\deg(h) > 1$, L. Migliorini, S. Kovács, E. Bedulev and the authors studied in [16], [12], [13], [14], [2], [24] [25] geometric properties of manifolds U mapping non-trivially to the moduli stack. Again, U can not be $\mathbb{C}^*$, nor an abelian variety, and more generally it must be Brody hyperbolic, if φ is quasi-finite.

In general the sheaf $\Omega^1_{X/Y}(\log S)$ fails to be big (see example 6.3). Nevertheless, building up on the methods introduced in [24] and [25] we will show that for m sufficiently large the sheaf $S^m(\Omega^1_Y(\log S))$ has enough global sections (see section 1 for the precise statement), to exclude the existence of a generically finite morphism $\varphi : U \to M_h$, or even of a non-trivial morphism, for certain manifolds U.

Theorem 0.1 (see 5.2, 5.3 and 7.2). *Assume that U satisfies one of the following conditions*

a) *U has a smooth projective compactification Y with $S = Y \setminus U$ a normal crossing divisor and with $T_Y(-\log S)$ weakly positive.*

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b) *Let $H_1+\cdots+H_\ell$ be a reduced normal crossing divisor in $\mathbb{P}^N$, and $\ell<\frac{N}{2}$. For $0\leq r\leq l$ define*

$$H=\bigcap_{j=r+1}^{\ell} H_j,\quad S_i=H_i|_H,\quad S=\sum_{i=1}^{r} S_i,$$

and assume $U=H\setminus S$.

c) *$U=\mathbb{P}^N\setminus S$ for a reduced normal crossing divisor $S=S_1+\cdots+S_\ell$ in $\mathbb{P}^N$, with $\ell<N$.*

Then a morphism $U\to M_h$, induced by a family, must be trivial.

In a) the sheaf $T_Y(-\log S)$ denotes the dual of the sheaf of one forms with logarithmic poles along S. The definition of "weakly positive" will be recalled in 1.1. Part a) of 0.1, for $S=\emptyset$, has been shown by S. Kovács in [14].

Considering $r=0$ in 0.1, b), one finds that smooth complete intersections U in $\mathbb{P}^N$ of codimension $\ell<\frac{N}{2}$ do not allow a non-trivial morphism $U\to M_h$, induced by a family. In b) the intersection with an empty index set is supposed to be $H=\mathbb{P}^N$. So for $\ell<\frac{N}{2}$ part c) follows from b).

In general, 0.1, c), will follow from the slightly stronger statement in the second part of the next theorem. In fact, if one chooses general linear hyperplanes $D_0,\cdots,D_N$ then $D_0+D_1+\cdots+D_N+S$ remains a normal crossing divisor. and all morphism

$$\mathbb{P}^N\setminus(D_0+D_1+\cdots+D_N+S)\longrightarrow M_h,$$

induced by a family, must be trivial.

Theorem 0.2 (see 7.2)**.** a) *Assume that U is the complement of a normal crossing divisor S with strictly less than N components in an N-dimensional abelian variety. Then there exists no generically finite morphism $U\to M_h$, induced by a family.*

b) *For $Y=\mathbb{P}^{\nu_1}\times\cdots\times\mathbb{P}^{\nu_k}$ let*

$$D^{(\nu_i)}=D_0^{(\nu_i)}+\cdots+D_{\nu_i}^{(\nu_i)}$$

be coordinate axes in $\mathbb{P}^{\nu_i}$ and

$$D=\bigoplus_{i=1}^{k} D^{(\nu_i)}.$$

Assume that $S=S_1+\cdots S_\ell$ is a divisor, such that $D+S$ is a reduced normal crossing divisor, and $\ell<\dim(Y)$. Then there exists no morphism $\varphi:U=Y\setminus(D+S)\to M_h$ with

$$\dim(\varphi(U))>\mathrm{Max}\{\dim(Y)-\nu_i;\ i=1,\ldots,k\}.$$

We do not know whether the bound $\ell<\dim(Y)$ in 0.2 is really needed. If the infinitesimal Torelli theorem holds true for the general fibre, hence if the family $V\to U$ induces a generically finite map to a period domain, then the fundamental group of U should not be abelian. In particular U can not be the complement of a normal crossing divisor in $\mathbb{P}^N$.

In section 6 we will prove different properties of U in case there exists a quasi-finite morphism $\varphi:U\to M_h$. Those properties will be related to the rigidity of generic curves in moduli stacks.

Theorem 0.3 (see 6.4 and 6.7)**.** *Let U be a quasi-projective variety and let $\varphi : U \to M_h$ be a quasi-finite morphism, induced by a family. Then*

a) *U can not be isomorphic to the product of more than $n = \deg(h)$ varieties of positive dimension.*
b) $\mathrm{Aut}(U)$ *is finite.*

Although we do not need it in its full strength, we could not resist to include a proof of the finiteness theorem 6.2, saying that for a projective curve C, for an open sub curve C_0, and for a projective compactification $\bar{U}$ of U, the morphisms $\pi : C \to \bar{U}$ with $\pi(C_0) \subset U$ are parameterized by a scheme of finite type.

We call $f : V \to U$ a (flat or smooth) family of projective varieties, if f is projective (flat or smooth) and all fibres connected. For a flat family, an invertible sheaf $\mathcal{L}$ on V will be called f-semi-ample, or relatively semi-ample over U, if for some $\nu > 0$ the evaluation of sections $f^* f_* \mathcal{L}^\nu \to \mathcal{L}^\nu$ is surjective. The notion f-ampleness will be used if in addition for $\nu \gg 0$ the induced U-morphism $V \to \mathbb{P}(f_* \mathcal{L}^\nu)$ is an embedding, or equivalently, if the restriction of $\mathcal{L}$ to all the fibres is ample.

For families over a higher dimensional base $f : V \to U$, the non-isotriviality will be measured by an invariant, introduced in [20]. We define $\mathrm{Var}(f)$ to be the smallest integer η for which there exists a finitely generated subfield K of $\overline{\mathbb{C}(U)}$ of transcendence degree η over $\mathbb{C}$, a variety F' defined over K, and a birational equivalence

$$V \times_U \mathrm{Spec}(\overline{\mathbb{C}(U)}) \sim F' \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{\mathbb{C}(U)}).$$

We will call f isotrivial, in case that $\mathrm{Var}(f) = 0$. If $(f : V \to U) \in \mathcal{M}_h(U)$ induces the morphism $\varphi : U \to M_h$, then $\mathrm{Var}(f) = \dim(\varphi(U))$.

Most of the results in this article carry over to families $V \to U$ with $\omega_{V/U}$ semi-ample. The first result without requiring local Torelli theorems, saying that there are no non-isotrivial families of elliptic surfaces over $\mathbb{C}^*$ or over elliptic curves, has been shown by K. Oguiso and the first named author [17]. It was later extended to all families of higher dimensional minimal models in [24].

Variant 0.4. *Let U be a quasi-projective manifold as in 0.1 or in 0.2. Then there exists no smooth family $f : V \to U$ with $\omega_{V/U}$ f-semi-ample and with* $\mathrm{Var}(f) = \dim(U)$.

Variant 0.5. *For U a quasi-projective manifold let $f : V \to U$ be a smooth family with $\omega_{V/U}$ f-semi-ample and with* $\mathrm{Var}(f) = \dim(U)$. *Then the conclusion a) and b) in 0.3 hold true.*

All the results mentioned will be corollaries of theorem 1.4, formulated in the first section. It is closely related to some conjectures and open problems on differential forms on moduli stacks, explained in 1.5. The proof of 1.4, which covers sections 2, 3, and 4, turns out to be quite complicated, and we will try to give an outline at the end of the first section.

The methods are close in spirit to the ones used in [24] for Y a curve, replacing [24], Proposition 1.3, by [26], Theorem 0.1, and using some of the tools developed in [25]. So the first three and a half sections do hardly contain any new ideas. They are needed nevertheless to adapt methods and notations to the situation studied here, and hopefully they can serve as a reference for methods needed to study positivity problems over higher dimensional bases. The reader who just wants to get some idea on the geometry of moduli stacks should skip sections 2, 3 and 4 in a first reading and start with sects 1, 5, 6 and 7.

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1. Differentialforms on moduli stacks

Our motivation and starting point are conjectures and questions on the sheaf of differential forms on moduli-stacks. Before formulating the technical main result and related conjectures and questions, let us recall some definitions.

Definition 1.1. Let $\mathcal{F}$ be a torsion free coherent sheaf on a quasi-projective normal variety Y and let $\mathcal{H}$ be an ample invertible sheaf.

a) $\mathcal{F}$ is generically generated if the natural morphism
$$H^0(Y,\mathcal{F}) \otimes \mathcal{O}_Y \longrightarrow \mathcal{F}$$
is surjective over some open dense subset U_0 of Y. If one wants to specify U_0 one says that $\mathcal{F}$ is globally generated over U_0.

b) $\mathcal{F}$ is weakly positive if there exists some dense open subset U_0 of Y with $\mathcal{F}|_{U_0}$ locally free, and if for all $\alpha > 0$ there exists some $\beta > 0$ such that
$$S^{\alpha\cdot\beta}(\mathcal{F}) \otimes \mathcal{H}^{\beta}$$
is globally generated over U_0. We will also say that $\mathcal{F}$ is weakly positive over U_0, in this case.

c) $\mathcal{F}$ is big if there exists some open dense subset U_0 in Y and some $\mu > 0$ such that
$$S^{\mu}(\mathcal{F}) \otimes \mathcal{H}^{-1}$$
is weakly positive over U_0. Underlining the role of U_0 we will also call $\mathcal{F}$ ample with respect to U_0.

Here, as in [20] and [25], we use the following convention: If $\mathcal{F}$ is a coherent torsion free sheaf on a quasi-projective normal variety Y, we consider the

largest open subscheme $i : Y_1 \to Y$ with $i^*\mathcal{F}$ locally free. For

$$\Phi = S^\mu, \quad \Phi = \bigotimes^{\mu} \quad \text{or} \quad \Phi = \det$$

we define

$$\Phi(\mathcal{F}) = i_*\Phi(i^*\mathcal{F}).$$

Let us recall two simple properties of sheaves which are ample with respect to open sets, or generically generated. A more complete list of such properties can be found in [23], §2. First of all the ampleness property can be expressed in a different way (see [25], 3.2, for example).

Lemma 1.2. *Let $\mathcal{H}$ be an ample invertible sheaf, and $\mathcal{F}$ a coherent torsion free sheaf on Y, whose restriction to some open dense subset $U_0 \subset Y$ is locally free. Then $\mathcal{F}$ is ample with respect to U_0 if and only if for some $\eta > 0$ there exists a morphism*

$$\bigoplus \mathcal{H} \longrightarrow S^\eta(\mathcal{F}),$$

surjective over U_0.

We will also need the following well known property of generically generated sheaves.

Lemma 1.3. *Let $\psi : Y' \to Y$ be a finite morphism and let $\mathcal{F}$ be a coherent torsion free sheaf on Y such that $\psi^*\mathcal{F}$ is generically generated. Then for some $\beta > 0$, the sheaf $S^\beta(\mathcal{F})$ is generically generated.*

Proof. We may assume that $\mathcal{F}$ is locally free, and replacing Y' by some covering, that Y' is a Galois cover of Y with Galois group G. Let

$$\pi : \mathbb{P} = \mathbb{P}(\mathcal{F}) \longrightarrow Y \quad \text{and} \quad \pi' : \mathbb{P}' = \mathbb{P}(\psi^*\mathcal{F}) \longrightarrow Y'$$

be the projective bundles. The induced covering $\psi' : \mathbb{P}' \to \mathbb{P}$ is again Galois. By assumption, for some $U_0 \subset Y$ the sheaf $\mathcal{O}_{\mathbb{P}'}(1)$ is generated by global sections over ${\psi'}^{-1}\pi^{-1}(U_0)$. Hence for $g = \#G$ the sheaf $\mathcal{O}_{\mathbb{P}'}(g) = {\psi'}^*\mathcal{O}_{\mathbb{P}}(g)$ is generated over ${\psi'}^{-1}\pi^{-1}(U_0)$ by G-invariant sections, hence $\mathcal{O}_{\mathbb{P}}(g)$ is globally generated by sections $s_1, \ldots, s_\ell \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(g))$ over $\pi^{-1}(U_0)$. By the Nullstellensatz, there exists some β' such that

$$S^{\beta'}\Big(\bigoplus_{i=1}^{\ell} \mathcal{O}_{\mathbb{P}} \cdot s_i\Big) \to S^{g\cdot\beta'}(\mathcal{F}) = \pi_*\mathcal{O}_{\mathbb{P}}(g \cdot \beta')$$

is surjective over U_0. $\square$

The main result of this article says, that the existence of smooth families $F : V \to U$ with $\mathrm{Var}(f) > 0$ is only possible if U carries multi-differential forms with logarithmic singularities at infinity.

Theorem 1.4. *Let Y be a projective manifold, S a reduced normal crossing divisor, and let $f : V \to U = Y \setminus S$ be a smooth family of n-dimensional projective varieties.*

i) *If $\omega_{V/U}$ is f-ample, then for some $m > 0$ the sheaf $S^m(\Omega^1_Y(\log S))$ contains an invertible sheaf $\mathcal{A}$ of Kodaira dimension $\kappa(\mathcal{A}) \geq \mathrm{Var}(f)$.*

ii) *If* $\omega_{V/U}$ *is* f*-ample and* $\mathrm{Var}(f) = \dim(Y)$*, then for some* $0 < m \leq n$ *the sheaf* $S^m(\Omega^1_Y(\log S))$ *contains a big coherent subsheaf* $\mathcal{P}$*.*
iii) *If* $\omega_{V/U}$ *is* f*-semi-ample and* $\mathrm{Var}(f) = \dim(Y)$*, then for some* $m > 0$ *the sheaf* $S^m(\Omega^1_Y(\log S))$ *contains a big coherent subsheaf* $\mathcal{P}$*.*
iv) *Moreover under the assumptions made in iii) there exists a non-singular finite covering* $\psi : Y' \to Y$ *and, for some* $0 < m \leq n$*, a big coherent subsheaf* $\mathcal{P}'$ *of* $\psi^* S^m(\Omega^1_Y(\log S))$*.*

Before giving a guideline to the proof of 1.4, let us discuss further properties of the sheaf of one forms on U, we hope to be true.

Problem 1.5. Let Y be a projective manifold, S a reduced normal crossing divisor, and $U = Y \setminus S$. Let $\varphi : U \to M_h$ be a morphism, induced by a family $f : V \to U$. Assume that the family $f : V \to U$ induces an étale map to the moduli stack, or in down to earth terms, that the induced Kodaira Spencer map

$$T_U \longrightarrow R^1 f_* T_{V/U}$$

is injective and locally split.

a) Is $\Omega^1_Y(\log S)$ weakly positive, or perhaps even weakly positive over U?
b) Is $\det(\Omega^1_Y(\log S)) = \omega_Y(S)$ big?
c) Are there conditions on Ω^1_F, for a general fibre F of f, which imply that $\Omega^1_Y(\log S)$ is big?

As we will see in 5.1, theorem 1.4 implies that the bigness in 1.5, b), follows from the weak positivity in a).

There is hope, that the questions a) and b), which have been raised by the first named author some time ago, will have an affirmative answer. In particular they have been verified by the second named author [26], under the additional assumption that the local Torelli theorem holds true for the general fibre F of f. The Brody hyperbolicity of moduli stacks of canonically polarized manifolds, shown in [25], the results of Kovács, and the content of this paper strengthen this hope. As S. Kovács told us, for certain divisors S in $Y = \mathbb{P}^N$, 1.5, a), holds true.

For moduli spaces of curves the sheaf $\Omega^1_Y(\log S)$ is ample with respect to U. This implies that morphisms $\pi : C_0 \to U$ are rigid (see 6.6). In the higher dimensional case the latter obviously does not hold true (see 6.3), and problem c) asks for conditions implying rigidity.

There is no evidence for the existence of a reasonable condition in c). One could hope that "Ω^1_F ample" or "Ω^1_F big" will work. At least, this excludes the obvious counter examples for the ampleness of $\Omega^1_Y(\log S)$, discussed in 6.3. For a non-isotrivial smooth family $V \to U$ of varieties with Ω^1_F ample, the restriction of Ω^1_V to F is big, an observation which for families of curves goes back to H. Grauert [9]. The problem 1.1, c), expresses our hope that such properties of global multi-differential forms on the general fibre could be mirrored in global properties of moduli spaces.

Notations. To prove 1.4 we start by choosing any non-singular projective compactification X of V, with $\Delta = X \setminus V$ a normal crossing divisor, such that $V \to U$ extends to a morphism $f : X \to Y$. For the proof of 1.4 we are

allowed to replace Y by any blowing up, if the pullback of S remains a normal crossing divisor. Moreover, as explained in the beginning of the next section, we may replace Y by the complement of a codimension two subscheme, and X by the corresponding preimage, hence to work with partial compactifications, as defined in 2.1. By abuse of notations, such a partial compactification will again be denoted by $f : X \to Y$.

In the course of the argument we will be forced to replace the morphism f by some fibred product. We will try to keep the following notations. A morphism $f' : X' \to Y'$ will denote a pullback of f under a morphism, usually dominant, $Y' \to Y$, or a desingularization of such a pullback. The smooth parts will be denoted by $V \to U$ and $V' \to U'$, respectively. $f^r : X^r \to Y$ will denote the family obtained as the r-fold fibred product over Y, and $f^{(r)} : X^{(r)} \to Y$ will be obtained as a desingularization of X^r. Usually U_0 will denote an open dense subscheme of Y, and $\tilde{U}$ will be a blowing up of U.

At several places we need in addition some auxiliary constructions. In section 2 this will be a family $g : Z \to Y'$, dominating birationally $X' \to Y'$ and a specific model $g' : Z' \to Y'$. For curves C mapping to Y, the desingularization of the induced family will be $h : W \to C$, where again some $'$ is added whenever we have to consider a pullback family over some covering C' of C.

Finally, in section 4 $h : W \to Y$ will be a blowing up of $X \to Y$ and $g : Z \to Y$ will be obtained as the desingularization of a finite covering of W.

Outline of the proof of 1.4. Let us start with 1.4, iii). In section 3 we will formulate and recall certain positivity properties of direct image sheaves. In particular, by [20] the assumptions in iii) imply that $\det(f_*\omega^\nu_{X/Y})$ is big, for some $\nu \gg 2$. This in turn implies that the sheaf $f_*\omega^\nu_{X/Y}$ is big. In 3.9 we will extend this result to the slightly smaller sheaf $f_*(\Omega^n_{X/Y}(\log \Delta))^\nu$. Hence for an ample invertible sheaf $\mathcal{A}$ on Y and for some $\mu \gg 1$ the sheaf $S^\mu(f_*(\Omega^n_{X/Y}(\log \Delta))^\nu) \otimes \mathcal{A}^{-1}$ will be globally generated over an open dense subset U_0. Replacing $f : X \to Y$ by a partial compactification we will assume this sheaf to be locally free. Then, for ν sufficiently large and divisible, $\Omega^n_{X/Y}(\log \Delta))^{\nu\mu} \otimes f^*\mathcal{A}^{-1}$ will be globally generated over $f^{-1}(U_0)$, for some $U_0 \neq \emptyset$.

This statement is not strong enough. We will need that

$$\text{(1.5.1)} \qquad \Omega^n_{X/Y}(\log \Delta)^\nu \otimes f^*\mathcal{A}^{-\nu} \text{ is globally generated over } f^{-1}(U_0),$$

for some $\nu \gg 2$ and for some ample invertible sheaf $\mathcal{A}$. To this aim we replace in 3.9 the original morphism $f : X \to Y$ by the r-th fibred product $f^{(r)} : X^{(r)} \to Y$, for some $r \gg 1$.

The condition (1.5.1) will reappear in section 4 in (4.3.1). There we study certain Higgs bundles $\bigoplus F^{p,q}$. (4.3.1) will allow to show, that $\mathcal{A} \otimes \bigoplus F^{p,q}$ is contained in a Higgs bundles induced by a variation of Hodge structures. The latter is coming from a finite cyclic covering Z of X. The negativity theorem in [26] will finish the proof of 1.4 iii).

For each of the other cases in 1.4 we need some additional constructions, most of which are discussed in section 2. In ii) and iv) (needed to prove 0.3, a) we have to bound m by the fibre dimension, hence we are not allowed to

replace $f : X \to Y$ by the fibre product $f^{(r)} : X^{(r)} \to Y$. Instead we choose a suitable covering $Y' \to Y$, in such a way that the assumption (4.3.1) holds true over the covering. For i) we have to present the family $X' \to Y'$ as the pullback of a family of maximal variation.

In section 2 we also recall the weak semi-stable reduction theorem due to Abramovich and Karu [1] and some of its consequences. In particular it will allow to construct a generically finite dominant morphism $Y' \to Y$ such that for a desingularization of the pullback family $f' : X' \to Y'$ the sheaves $\bigotimes^{\mu} f'_* \omega^{\nu}_{X'/Y'}$ are reflexive. This fact was used in [25] in the proof of 3.9. As mentioned, we can restrict ourselves to partial compactifications $f : X \to Y$, and repeating the arguments from [25] in this case, we would not really need the weakly semi-stable reduction. However, the proof of the finiteness theorem 6.2 is based on this method.

As in [24] it should be sufficient in 1.4, iii) and iv), to require that the fibres F of f are of general type, or in the case $0 \leq \kappa(F) < \dim(F)$ that F is birational to some F' with $\omega_{F'}$ semi-ample. We do not include this, since the existence of relative base loci make the notations even more confusing than they are in the present version. However, comparing the arguments in [24], §3, with the ones used here, it should not be too difficult to work out the details.

2. Mild morphisms

As explained at the end of the last section it will be convenient, although not really necessary, to use for the proof of 1.4 some of the results and constructions contained in [25], in particular the weak semi-stable reduction theorem due to Abramovich and Karu [1]. It will allow us to formulate the strong positivity theorem 3.9 for product families, shown in [25], 4.1., and it will be used in the proof of the boundedness of the functor of homomorphism in 6.2. We will use it again to reduce the proof of 1.4, i), to the case of maximal variation, although this part could easily be done without the weak semi-stable reduction. We also recall Kawamata's covering construction. The content of this section will be needed in the proof of parts i), ii) and iv) of 1.4, but not for iii).

Definition 2.1.

a) Given a family $\tilde{V} \to \tilde{U}$ we will call $V \to U$ a birational model if there exist compatible birational morphisms $\tau : U \to \tilde{U}$ and $\tau' : V \to \tilde{V} \times_{\tilde{U}} U$. If we underline that U and $\tilde{U}$ coincide, we want τ to be the identity. If $\tilde{V} \to \tilde{U}$ is smooth, we call $V \to U$ a smooth birational model, if τ' is an isomorphism.

b) If $V \to U$ is a smooth projective family of quasi-projective manifolds, we call $f : X \to Y$ a partial compactification, if
   i) X and Y are quasi-projective manifolds, and $U \subset Y$.
   ii) Y has a non-singular projective compactification $\bar{Y}$ such that S extends to a normal crossing divisor and such that $\operatorname{codim}(\bar{Y} \setminus Y) \geq 2$.
   iii) f is a projective morphism and $f^{-1}(U) \to U$ coincides with $V \to U$.
   iv) $S = Y \setminus U$, and $\Delta = f^* S$ are normal crossing divisors.

c) We say that a partial compactification $f : X \to Y$ is a good partial compactification if the condition iv) in b) is replaced by
   iv) f is flat, $S = Y \setminus U$ is a smooth divisor, and $\Delta = f^*S$ is a relative normal crossing divisor, i.e. a normal crossing divisor whose components, and all their intersections are smooth over components of S.

d) The good partial compactification $f : X \to Y$ is semi-stable, if in c), iv), the divisor f^*S is reduced.

e) An arbitrary partial compactification of $V \to U$ is called semi-stable in codimension one, if it contains a semi-stable good partial compactification.

Remark 2.2. The second condition in b) or c) allows to talk about invertible sheaves $\mathcal{A}$ of positive Kodaira dimension on Y. In fact, $\mathcal{A}$ extends in a unique way to an invertible sheaf $\bar{\mathcal{A}}$ on $\bar{Y}$ and

$$H^0(Y, \mathcal{A}^\nu) = H^0(\bar{Y}, \bar{\mathcal{A}}^\nu),$$

for all ν. So we can write $\kappa(\mathcal{A}) := \kappa(\bar{\mathcal{A}})$, in case Y allows a compactification satisfying ii). If $\tau : \bar{Y}' \to \bar{Y}$ is a blowing up with centers in $\bar{Y} \setminus Y$, and if $\bar{\mathcal{A}}'$ is an extension of $\mathcal{A}$ to $\bar{Y}'$, then $\kappa(\mathcal{A}) \geq \kappa(\bar{\mathcal{A}}')$.

In a similar way, one finds a coherent sheaf $\mathcal{F}$ on Y to be semi-ample with respect to $U_0 \subset Y$ (or weakly positive over U_0), if and only if its extension to $\bar{Y}$ has the same property.

Kawamata's covering construction will be used frequently throughout this article. First of all, it allows the semi-stable reduction in codimension one, and secondly it allows to take roots out of effective divisors.

Lemma 2.3.

a) *Let Y be a quasi-projective manifold, S a normal crossing divisor, and let $\mathcal{A}$ be an invertible sheaf, globally generated over Y. Then for all μ there exists a non-singular finite covering $\psi : Y' \to Y$ whose discriminant $\Delta(Y'/Y)$ does not contain components of S, such that $\psi^*(S + \Delta(Y'/Y))$ is a normal crossing divisor, and such that $\psi^*\mathcal{A} = \mathcal{O}_{Y'}(\mu \cdot A')$ for some reduced non-singular divisor A' on Y'.*

b) *Let $f : X \to Y$ be a partial compactification of a smooth family $V \to U$. Then there exists a non-singular finite covering $\psi : Y' \to Y$, and a desingularization $\psi' : X' \to X \times_Y Y'$ such that the induced family $f' : X' \to Y'$ is semi-stable in codimension one.*

Proof. Given positive integers ϵ_i for all components S_i of S, Kawamata constructed a finite non-singular covering $\psi : Y' \to Y$ (see [23], 2.3), with $\psi^*(S + \Delta(Y'/Y))$ a normal crossing divisor, such that all components of ψ^*S_i are ramified of order exactly ϵ_i.

In a) we choose A to be the zero-divisor of a general section of $\mathcal{A}$, and we apply Kawamata's construction to $S + A$, where the ϵ_i are one for the components of S and where the prescribed ramification index for A is μ.

In b) the semi-stable reduction theorem for families over curves (see [11]) allows to choose the ϵ_i such that the family $f' : X' \to Y'$ is semi-stable in codimension one. $\square$

Unfortunately in 2.3, b), one has little control on the structure of f' over the singularities of S. Here the weak semi-stable reduction theorem will be of help. The pullback of a weakly semi-stable morphism under a dominant morphism of manifolds is no longer weakly semi-stable. However some of the properties of a weakly semi-stable morphism survive. Those are collected in the following definition, due again to Abramovich and Karu [1].

Definition 2.4. A projective morphism $g' : Z' \to Y'$ between quasi-projective varieties is called mild, if

a) g' is flat, Gorenstein with reduced fibres.
b) Y' is non-singular and Z' normal with at most rational singularities.
c) Given a dominant morphism $Y'_1 \to Y'$ where Y'_1 has at most rational Gorenstein singularities, $Z' \times_{Y'} Y'_1$ is normal with at most rational singularities.
d) Let Y'_0 be an open subvariety of Y', with ${g'}^{-1}(Y'_0) \to Y'_0$ smooth. Given a non-singular curve C' and a morphism $\pi : C' \to Y'$ whose image meets Y'_0, the fibred product $Z' \times_{Y'} C'$ is normal, Gorenstein with at most rational singularities.

The mildness of g' is, more or less by definition, compatible with pullback. Let us rephrase three of the properties shown in [25], 2.2.

Lemma 2.5. *Let Z and Y be quasi-projective manifolds, $g : Z \to Y'$ be a projective, birational to a projective mild morphism $g' : Z' \to Y'$. Then one has:*

i) *For all $\nu \geq 1$ the sheaf $g_*\omega^\nu_{Z/Y'}$ is reflexive and isomorphic to $g'_*\omega^\nu_{Z'/Y'}$.*
ii) *If $\gamma : Y'' \to Y'$ is a dominant morphism between quasi-projective manifolds, then the morphism $pr_2 : Z \times_{Y'} Y'' \to Y''$ is birational to a projective mild morphism to Y''.*
iii) *Let $Z^{(r)}$ be a desingularization of the r-fold fibre product $Z \times_{Y'} \cdots \times_{Y'} Z$. Then the induced morphism $Z^{(r)} \to Y'$ is birational to a projective mild morphism over Y'.*

One consequence of the weakly semi-stable reduction says that, changing the birational model of a morphism, one always finds a finite cover of the base such that the pullback is birational to a mild morphism (see [25], 2.3).

Lemma 2.6. *Let $V \to U$ be a smooth family of projective varieties. Then there exists a quasi-projective manifold $\tilde{U}$ and a smooth birational model $\tilde{V} \to \tilde{U}$, non-singular projective compactifications Y of $\tilde{U}$ and X of $\tilde{V}$, with $S = Y \setminus \tilde{U}$ and $\Delta = X \setminus \tilde{V}$ normal crossing divisors, and a diagram of projective morphisms*

(2.6.1)
$$\begin{tikzcd} X & X' \arrow[l, "\psi'"'] \arrow[d, "f'"] & Z \arrow[l, "\sigma"'] \arrow[dl, "g"] \\ Y \arrow[from=u, "f"] & Y' \arrow[l, "\psi"'] & \end{tikzcd}$$

with:

a) *Y' and Z are non-singular, X' is the normalization of $X \times_Y Y'$, and σ is a desingularization.*

b) *g is birational to a mild morphism $g' : Z' \to Y'$.*
c) *For all $\nu > 0$ the sheaf $g_*\omega^\nu_{Z/Y'}$ is reflexive and there exists an injection*

$$g_*\omega^\nu_{Z/Y'} \longrightarrow \psi^* f_*\omega^\nu_{X/Y}.$$

d) *For some positive integer N_ν, and for some invertible sheaf λ_ν on Y*

$$\det(g_*\omega^\nu_{Z/Y'})^{N_\nu} = \psi^*\lambda_\nu.$$

e) *Moreover, if $\bar{Y}$ is a given projective compactification of U, we can assume that there is a birational morphism $Y \to \bar{Y}$.*

This diagram has been constructed in [25], §2. Let us just recall that the reflexivity of $g_*\omega^\nu_{Z/Y'}$ in c) is a consequence of b), using 2.5, i). □

Unfortunately the way it is constructed, the mild model $g' : Z' \to Y'$ might not be smooth over $\psi^{-1}(\tilde{U})$. Moreover even in case U is non-singular one has to allow blowing ups $\tau : \tilde{U} \to U$. Hence starting from $V \to U$ we can only say that U contains some "good" open dense subset U_g for which τ is an isomorphism between $\tilde{U}_g := \tau^{-1}(U_g)$ and U_g, and for which

$$g'^{-1}\psi^{-1}(\tilde{U}_g) \longrightarrow \psi^{-1}(\tilde{U}_g)$$

is smooth. The next construction will be needed in the proof of 6.2.

Corollary 2.7. *Let C be a non-singular projective curve, $C_0 \subset C$ an open dense subset, and let $\pi_0 : C_0 \to U$ be a morphism with $\pi_0(C_0) \cap U_g \neq \emptyset$. Hence there is a lifting of π_0 to $\tilde{U}$ and an extension $\pi : C \to Y$ with $\pi_0 = \tau \circ \pi|_{C_0}$. Let $h : W \to C$ be the family obtained by desingularizing the main component of the normalization of $X \times_Y C$, and let λ_ν and N_ν be as in 2.6, d). Then*

$$\deg(\pi^*\lambda_\nu) \leq N_\nu \cdot \deg(\det(h_*\omega^\nu_{W/C})).$$

Proof. Let $\rho : C' \to C$ be a finite morphism of non-singular curves such that π lifts to $\pi' : C' \to Y'$, and let $h' : W' \to C'$ be the family obtained by desingularizing the main component of the normalization of $X' \times_{Y'} C'$. By condition d) in the definition of a mild morphism, and by the choice of U_g, the family h' has $Z' \times_{Y'} C'$ as a mild model. Applying 2.6, c), to h and h', and using 2.6, d), we find

$$\deg(\rho) \cdot \deg(\pi^*\lambda_\nu) = N_\nu \cdot \deg(\pi'^* g_*\omega^\nu_{Z/Y'}) \quad \text{and}$$
$$\deg(h'_*\omega^\nu_{W'/C'}) \leq \deg(\rho) \cdot \deg(h_*\omega^\nu_{W/C}).$$

Moreover, by 2.6, c), and by base change, one obtains a morphism of sheaves

$$\pi'^* g_*\omega^\nu_{Z/Y'} \simeq \pi'^* g'_*\omega^\nu_{Z'/Y'} \longrightarrow pr_{2*}\omega^\nu_{Z'\times_{Y'}C'/C'} \simeq h'_*\omega^\nu_{W'/C'}, \tag{2.7.1}$$

which is an isomorphism over some open dense subset. Let r denote the rank of those sheaves.

It remains to show, that (2.7.1) induces an injection from $\pi'^* \det(g'_*\omega^\nu_{Z'/Y'})$ into $\det(pr_{2*}\omega^\nu_{Z'\times_{Y'}C'/C'})$. To this aim, as in 2.5, iii), let "$^{(r)}$" stand for "taking a desingularization of the r-th fibre product". Then $g^{(r)} : Z^{(r)} \to Y'$ is again

birational to a mild morphism over Y'. As in [25], 4.1.1, flat base change and the projection formula give isomorphisms

$$h'^{(r)}_{*}\omega^{\nu}_{W'^{(r)}/C'} \simeq \bigotimes^{r} h'_{*}\omega^{\nu}_{W'/C'} \quad \text{and} \quad g^{(r)}_{*}\omega^{\nu}_{Z^{(r)}/Y'} \simeq \bigotimes^{r} g_{*}\omega^{\nu}_{Z/Y'},$$

the second one outside of a codimension two subscheme. Since both sheaves are reflexive, the latter extends to Y'.

The injection (2.7.1), applied to $g^{(r)}$ and $h^{(r)}$ induces an injective morphism

$$\pi'^{*}\bigotimes^{r} g_{*}\omega^{\nu}_{Z/Y'} \longrightarrow \bigotimes^{r} h'_{*}\omega^{\nu}_{W'/C'}. \tag{2.7.2}$$

The left hand side contains $\pi'^{*}\det(g'_{*}\omega^{\nu}_{Z'/Y'})$ as direct factor, whereas the right-hand contains $\det(pr_{2*}\omega^{\nu}_{Z'\times_{Y'}C'/C'})$, and we obtain the injection for the determinant sheaves as well. □

The construction of (2.6.1) in [25] will be used to construct a second diagram (2.8.1). There we do not insist on the projectivity of the base spaces, and we allow ourselves to work with good partial compactifications of an open subfamily of the given one. This quite technical construction will be needed to proof 1.4, i).

Lemma 2.8. *Let $V \to U$ be a smooth family of canonically polarized manifolds. Let $\bar{Y}$ and $\bar{X}$ be non-singular projective compactifications of U and V such that both, $\bar{Y}\setminus U$ and $\bar{X}\setminus V$, are normal crossing divisors and such that $V \to U$ extends to $\bar{f}:\bar{X}\to\bar{Y}$. Blowing up $\bar{Y}$ and $\bar{X}$, if necessary, one finds an open subscheme Y in $\bar{Y}$ with $\operatorname{codim}(\bar{Y}\setminus Y)\geq 2$ such that the restriction $f:Y\to X$ is a good partial compactification of a smooth birational model of $V\to U$, and one finds a diagram of morphisms between quasi-projective manifolds*

$$\begin{tikzcd} X \arrow[d, "f"] & X' \arrow[l, "\psi'"'] \arrow[d, "f'"] & Z \arrow[l, "\sigma"'] \arrow[dl, "g"] \arrow[r, "\eta'"] & Z^{\#} \arrow[dl, "g^{\#}"] \\ Y & Y' \arrow[l, "\psi"'] \arrow[r, "\eta"] & Y^{\#} & \end{tikzcd} \tag{2.8.1}$$

with:

a) *$g^{\#}$ is a projective morphism, birational to a mild projective morphism $g^{\#'}:Z^{\#'}\to Y^{\#}$.*
b) *$g^{\#}$ is semi-stable in codimension one.*
c) *$Y^{\#}$ is projective, η is dominant and smooth, η' factors through a birational morphism $Z\to Z^{\#}\times_{Y^{\#}}Y'$, and ψ is finite.*
d) *X' is the normalization of $X\times_Y Y'$ and σ is a blowing up with center in $f'^{-1}\psi^{-1}(S')$. In particular f' and g' are projective.*
e) *Let $U^{\#}$ be the largest subscheme of $Y^{\#}$ with*
$$V^{\#}=g^{\#^{-1}}(U^{\#})\longrightarrow U^{\#}$$
smooth. Then $\psi^{-1}(U)\subset\eta^{-1}(U^{\#})$, and $U^{\#}$ is generically finite over M_h.
f) *For all $\nu>0$ there are isomorphisms*
$$g_{*}\omega^{\nu}_{Z/Y'}\simeq\eta^{*}g^{\#}_{*}\omega^{\nu}_{Z^{\#}/Y^{\#}} \quad \textit{and} \quad \det(g_{*}\omega^{\nu}_{Z/Y'})\simeq\eta^{*}\det(g^{\#}_{*}\omega^{\nu}_{Z^{\#}/Y^{\#}}).$$

g) *For all $\nu > 0$ there exists an injection*

$$g_*(\omega^\nu_{Z/Y'}) \longrightarrow \psi^* f_*(\omega^\nu_{X/Y}).$$

For some positive integer N_ν, and for some invertible sheaf λ_ν on Y

$$\det(g_*(\omega^\nu_{Z/Y'}))^{N_\nu} = \psi^*\lambda_\nu.$$

Proof. It remains to verify, that the construction given in [25], §2, for (2.6.1) can be modified to guaranty the condition e) along with the others.

Let $\varphi : U \to M_h$ be the induced morphism to the moduli scheme. Seshadri and Kollár constructed a finite Galois cover of the moduli space which is induced by a family (see [23], 9.25, for example). Hence there exists some manifold $U^\#$, generically finite over the closure of $\varphi(U)$ such that the morphism $U^\# \to M_h$ is induced by a family $V^\# \to U^\#$. By [25], 2.3, blowing up $U^\#$, if necessary, we find a projective compactification $Y^\#$ of $U^\#$ and a covering $Y^{\#'}$, such that

$$V^\# \times_{Y^\#} Y^{\#'} \longrightarrow Y^{\#'}$$

is birational to a projective mild morphism over $Y^{\#'}$. Replacing $U^\#$ by some generically finite cover, we can assume that $V^\# \to U^\#$ has such a model already over $Y^\#$.

Next let Y' be any variety, generically finite over $\bar{Y}$, for which there exists a morphism $\eta : Y' \to Y^\#$. By 2.5, ii), we are allowed to replace $Y^\#$ by any manifold, generically finite over $Y^\#$, without loosing the mild birational model. Doing so, we can assume the fibres of $Y' \to Y^\#$ to be connected. Replacing Y' by some blowing up, we may assume that for some non-singular blowing up $Y \to \bar{Y}$ the morphism $Y' \to \bar{Y}$ factors through a finite morphism $Y' \to Y$.

Next choose a blowing up $Y^{\#'} \to Y^\#$ such that the main component of $Y' \times_{Y^\#} Y^{\#'}$ is flat over $Y^\#$, and Y'' to be a desingularization. Hence changing notations again, and dropping one prime, we can assume that the image of the largest reduced divisor E in Y' with $\operatorname{codim}(\eta(E)) \geq 2$ maps to a subscheme of Y of codimension larger that or equal to 2. This remains true, if we replace $Y^\#$ and Y' by finite coverings. Applying 2.3, b), to $Y' \to Y^\#$, provides us with non-singular covering of $Y^\#$ such that a desingularization of the pullback of $Y' \to Y^\#$ is semi-stable in codimension one. Again, this remains true if we replace $Y^\#$ by a larger covering, and using 2.3, b), a second time, now for $Z^\# \to Y^\#$, we can assume that this morphism is as well semi-stable in codimension one.

Up to now, we succeeded to find the manifolds in (2.8.1) such that a) and b) hold true. In c), the projectivity of $Y^\#$ and the dominance of Y' over $Y^\#$ follow from the construction. For the divisor E in Y' considered above, we replace Y by $Y \setminus \psi(E)$ and Y' by $Y' \setminus E$, and of course X, X' and Z by the corresponding preimages. Then the non-equidimensional locus of η in Y' will be of codimension larger than or equal to two. ψ is generically finite, by construction, hence finite over the complement of a codimension two subscheme of Y. Replacing again Y by the complement of codimension two subscheme, we can assume η to be equidimensional, hence flat, and ψ to be finite. The morphism η has reduced fibres over general points of divisors in $Y^\#$, hence

it is smooth outside a codimension two subset of Y', and replacing Y by the complement of its image, we achieved c).

Since $V \to U$ is smooth, the pullback of $X \to Y$ to Y' is smooth outside of $\psi^{-1}(S)$. Moreover the induced morphism to the moduli scheme M_h factors through an open subset of $Y^{\#}$. Since by construction $U^{\#}$ is proper over its image in M_h, the image of $\psi^{-1}(U)$ lies in $U^{\#}$ and we obtain d) and e).

For f) remark that the pullback $Z' \to Y'$ of the mild projective morphism $g^{\#} : Z^{\#'} \to Y^{\#}$ to Y' is again mild, and birational to $Z \to Y'$. By flat base change,

$$g'_*\omega^{\nu}_{Z'/Y'} \simeq \eta^* {g^{\#}}'_* \omega^{\nu}_{Z^{\#'}/Y^{\#}}.$$

Since Z' and $Z^{\#'}$ are normal with at most rational Gorenstein singularities, we obtain (as in [25], 2.3) that the sheaf on the right hand side is $g_*\omega^{\nu}_{Z/Y'}$ whereas the one on the right hand side is $\eta^* {g^{\#}}_* \omega^{\nu}_{Z^{\#}/Y^{\#}}$.

g) coincides with 2.6, d), and it has been verified in [25], 2.4. as a consequence of the existence of a mild model for g over Y'. □

3. Positivity and ampleness

Next we will recall positivity theorems, due to Fujita, Kawamata, Kollár and the first named author. Most of the content of this section is well known, or easily follows from known results.

As in 1.4 we will assume throughout this section that U is the complement of a normal crossing divisor $\bar{S}$ in a manifold $\bar{Y}$, and that there is a smooth family $V \to U$ with $\omega_{V/U}$ relative semi-ample. Leaving out a codimension two subset in $\bar{Y}$ we find a good partial compactification $f : X \to Y$, as defined in 2.1.

For an effective $\mathbb{Q}$-divisor $D \in \mathrm{Div}(X)$ the integral part $[D]$ is the largest divisor with $[D] \leq D$. For an effective divisor Γ on X, and for $N \in \mathbb{N} - \{0\}$ the algebraic multiplier sheaf is

$$\omega_{X/Y}\Big\{\frac{-\Gamma}{N}\Big\} = \psi_*\Big(\omega_{T/Y}\Big(-\Big[\frac{\Gamma'}{N}\Big]\Big)\Big)$$

where $\psi : T \to X$ is any blowing up with $\Gamma' = \psi^*\Gamma$ a normal crossing divisor (see for example [6], 7.4, or [23], section 5.3).

Let F be a non-singular fibre of f. Using the definition given above for F, instead of X, and for a divisor Π on F, one defines

$$e(\Pi) = \mathrm{Min}\Big\{N \in \mathbb{N} \setminus \{0\};\ \omega_F\Big\{\frac{-\Pi}{N}\Big\} = \omega_F\Big\}.$$

By [6] or [23], section 5.4, $e(\Gamma|_F)$ is upper semi-continuous, and there exists a neighborhood V_0 of F with $e(\Gamma|_{V_0}) \leq e(\Gamma|_F)$. If $\mathcal{L}$ is an invertible sheaf on F, with $H^0(F, \mathcal{L}) \neq 0$, one defines

$$e(\mathcal{L}) = \mathrm{Max}\big\{e(\Pi);\ \Pi \text{ an effective divisor and } \mathcal{O}_F(\Pi) = \mathcal{L}\big\}.$$

Proposition 3.1 ([25], 3.3)**.** *Let $\mathcal{L}$ be an invertible sheaf, let Γ be a divisor on X, and let $\mathcal{F}$ be a coherent sheaf on Y. Assume that, for some $N > 0$ and for some open dense subscheme U_0 of U, the following conditions hold true:*

a) *$\mathcal{F}$ is weakly positive over U_0 (in particular $\mathcal{F}|_{U_0}$ is locally free).*

b) *There exists a morphism* $f^*\mathcal{F} \to \mathcal{L}^N(-\Gamma)$*, surjective over* $f^{-1}(U_0)$.

c) *None of the fibres* F *of* $f : V_0 = f^{-1}(U_0) \to U_0$ *is contained in* Γ*, and for all of them*

$$e(\Gamma|_F) \leq N.$$

Then $f_*(\mathcal{L} \otimes \omega_{X/Y})$ *is weakly positive over* U_0.

As mentioned in [25], 3.8, the arguments used in [23], 2.45, carry over to give a simple proof of the following, as a corollary of 3.1.

Corollary 3.2 ([22], 3.7)**.** $f_*\omega^\nu_{X/Y}$ *is weakly positive over* U.

In [21], for families of canonically polarized manifolds and in [10], in general, one finds the strong positivity theorem saying:

Theorem 3.3. *If* $\omega_{V/U}$ *is* f*-semi-ample, then for some* η *sufficiently large and divisible,*

$$\kappa(\det(f_*\omega^\eta_{X/Y})) \geq \mathrm{Var}(f).$$

In case $\mathrm{Var}(f) = \dim(Y)$, 3.3 implies that $\det(f_*\omega^\eta_{X/Y})$ is ample with respect to some open dense subset U_0 of Y. If the general fibre of f is canonically polarized, and if the induced map $\varphi : U \to M_h$ quasi-finite over its image, one can choose $U_0 = U$, as follows from the last part of the next proposition.

Proposition 3.4. *Assume that* $\mathrm{Var}(f) = \dim(Y)$*, and that* $\omega_{V/U}$ *is* f*-semi-ample. Then:*

i) *The sheaf* $f_*\omega^\nu_{X/Y}$ *is ample with respect to some open dense subset* U_0 *of* Y *for all* $\nu > 1$ *with* $f_*\omega^\nu_{X/Y} \neq 0$.

ii) *If* B *is an effective divisor, supported in* S *then for all* ν *sufficiently large and divisible, the sheaf* $\mathcal{O}_Y(-B) \otimes f_*\omega^\nu_{X/Y}$ *is ample with respect to some open dense subset* U_0.

iii) *If the smooth fibres of* f *are canonically polarized and if the induced morphism* $\varphi : U \to M_h$ *is quasi-finite over its image, then one can chose* $U_0 = U$ *in i) and ii).*

Proof. For iii) one uses a variant of 3.3, which has been shown [22], 1.19. It also follows from the obvious extension of the ampleness criterion in [23], 4.33, to the case "ample with respect to U":

Claim 3.5. Under the assumption made in 3.4, iii), for all η sufficiently large and divisible, there exist positive integers a, b and μ such that

$$\det(f_*\omega^{\mu\eta}_{X/Y})^a \otimes \det(f_*\omega^\eta_{X/Y})^b$$

ample with respect to U. □

Since we do not want to distinguish between the two cases i) and iii) in 1.4, we choose $U_0 = U$ in iii), and we allow $a = \mu = 0$ in case i). By 3.5 and 3.3, respectively, in both cases the sheaf $\det(f_*\omega^{\mu\eta}_{X/Y})^a \otimes \det(f_*\omega^\eta_{X/Y})^b$ is ample with respect to U_0.

By [6], §7, or [23], Section 5.4, the number $e(\omega^{\mu\eta}_F)$ is bounded by some constant e, for all smooth fibres of f. We will choose e to be divisible by η and larger than $\mu\eta$.

Replacing a and b by some multiple, we may assume that there exists a very ample sheaf $\mathcal{A}$ and a morphism

$$\mathcal{A} \longrightarrow \det(f_*\omega_{X/Y}^{\mu\eta})^a \otimes \det(f_*\omega_{X/Y}^{\eta})^b$$

which is an isomorphism over U_0, and that b is divisible by μ.

By 2.3, a), there exists a non-singular covering $\psi : Y' \to Y$ and an effective divisor H with $\psi^*\mathcal{A} = \mathcal{O}_{Y'}(e \cdot (\nu - 1) \cdot H)$, and such that the discriminant locus $\Delta(Y'/Y)$ does not contain any of the components of S. Replacing Y by a slightly smaller scheme, we can assume that $\Delta(Y'/Y) \cap S = \emptyset$, hence $X' = X \times_Y Y'$ is non-singular and by flat base change

$$pr_{2*}\omega_{X'/Y'}^{\sigma} = \psi^* f_*\omega_{X/Y}^{\sigma}$$

for all σ. The assumptions in 3.4, i), ii) or iii) remain true for $pr_2 : X' \to Y'$, and by [23], 2.16, it is sufficient to show that the conclusions in 3.4 hold true on Y' for $\psi^{-1}(U_0)$.

Dropping the primes, we will assume in the sequel that $\mathcal{A}$ has a section whose zero-divisor is $e \cdot (\nu - 1) \cdot H$ for a non-singular divisor H.

Let $r(\sigma)$ denote the rank of $f_*\omega_{X/Y}^{\sigma}$. We choose

$$r = r(\eta) \cdot \frac{b}{\mu} + r(\mu\eta) \cdot a,$$

consider the r-fold fibre product

$$f^r : X^r = X \times_Y X \ldots \times_Y X \longrightarrow Y,$$

and a desingularization $\delta : X^{(r)} \to X^r$. Using flat base change, and the natural maps

$$\mathcal{O}_{X^r} \longrightarrow \delta_*\mathcal{O}_{X^{(r)}} \quad \text{and} \quad \delta_*\omega_{X^{(r)}} \longrightarrow \omega_{X^r},$$

one finds morphisms

$$\bigotimes^r f_*\omega_{X/Y}^{\mu\eta} \longrightarrow f_*^{(r)}\delta^*\omega_{X^r/Y}^{\mu\eta} \qquad \text{and} \tag{3.5.1}$$

$$f_*^{(r)}\delta^*(\omega_{X^r/Y}^{\nu-1} \otimes \omega_{X^{(r)}/Y}) \longrightarrow f_*^r\omega_{X^r/Y}^{\nu} = \bigotimes^r f_*\omega_{X/Y}^{\nu}, \tag{3.5.2}$$

and both are isomorphism over U. We have natural maps

$$\det(f_*\omega_{X/Y}^{\mu\eta}) \xrightarrow{\subset} \bigotimes^{r(\mu\eta)} f_*\omega_{X/Y}^{\mu\eta} \quad \text{and} \tag{3.5.3}$$

$$\det(f_*\omega_{X/Y}^{\eta})^{\mu} \xrightarrow{\subset} \bigotimes^{r(\eta)\cdot\mu} f_*\omega_{X/Y}^{\eta} \longrightarrow \bigotimes^{r(\eta)} f_*\omega_{X/Y}^{\mu\eta}, \tag{3.5.4}$$

where the last morphism is the multiplication map. Hence we obtain

$$\mathcal{A} = \mathcal{O}_Y(e \cdot (\nu - 1) \cdot H) \longrightarrow \det(f_*\omega_{X/Y}^{\mu\eta})^a \otimes \det(f_*\omega_{X/Y}^{\eta})^b \longrightarrow$$
$$\bigotimes^r f_*\omega_{X/Y}^{\mu\eta} \longrightarrow f_*^{(r)}\delta^*\omega_{X^r/Y}^{\mu\eta}.$$

Thereby the sheaf ${f^{(r)}}^*\mathcal{A}$ is a subsheaf of $\delta^*\omega_{X^r/Y}^{\mu\eta}$. Let Γ be the zero divisor of the corresponding section of

$$f^{(r)*}\mathcal{A}^{-1} \otimes \delta^*\omega_{X^r/Y}^{\mu\eta} \quad \text{hence} \quad \mathcal{O}_{X^{(r)}}(-\Gamma) = f^{(r)*}\mathcal{A} \otimes \delta^*\omega_{X^r/Y}^{-\mu\eta}.$$

For the sheaf
$$\mathcal{M} = \delta^*(\omega_{X^r/Y} \otimes \mathcal{O}_{X^r}(-f^{r*}H))$$
one finds
$$\mathcal{M}^{e\cdot(\nu-1)}(-\Gamma) = \delta^*\omega_{X^r/Y}^{e\cdot(\nu-1)} \otimes f^{(r)*}\mathcal{A}^{-1} \otimes \mathcal{O}_{X^{(r)}}(-\Gamma) = \delta^*\omega_{X^r/Y}^{e\cdot(\nu-1)-\mu\eta}.$$
By the assumption 3.4, i), and by the choice of e we have a morphism
$$f^*f_*\omega_{X/Y}^{e\cdot(\nu-1)-\mu\eta} \longrightarrow \omega_{X/Y}^{e\cdot(\nu-1)-\mu\eta},$$
surjective over $f^{-1}(U_0)$. The sheaf
$$\mathcal{F} = \bigotimes^r f_*\omega_{X/Y}^{e\cdot(\nu-1)-\mu\eta}$$
is weakly positive over U_0 and there is a morphism
$$f^{(r)*}\mathcal{F} \longrightarrow \mathcal{M}^{e\cdot(\nu-1)}(-\Gamma)$$
surjective over $f^{-1}(U_0)$. Since the morphism of sheaves in (3.5.3), as well as the first one in (3.5.4), split locally over U_0 the divisor Γ can not contain a fibre F of
$${f^{(r)}}^{-1}(U_0) \longrightarrow U_0,$$
and by [6], §7, or [23], 5.21, for those fibres
$$e(\Gamma|_{F^r}) \leq e(\omega_{F^r}^{\mu\eta}) = e(\omega_F^{\mu\eta}) \leq e.$$
Applying 3.1 to $\mathcal{L} = \mathcal{M}^{\nu-1}$ one obtains the weak positivity of the sheaf
$$f_*^{(r)}(\mathcal{M}^{\nu-1} \otimes \omega_{X^{(r)}/Y}) = f_*^{(r)}(\delta^*(\omega_{X^r/Y}^{\nu-1} \otimes \omega_{X^{(r)}/Y})) \otimes \mathcal{O}_Y(-(\nu-1)\cdot H)$$
over U_0. By (3.5.2) one finds morphisms, surjective over U_0
$$\begin{aligned} &f_*^{(r)}(\delta^*(\omega_{X/Y}^{\nu-1} \otimes \omega_{X^{(r)}/Y})) \otimes \mathcal{O}_Y(-(\nu-1)\cdot H) \\ &\longrightarrow f_*^r(\omega_{X^r/Y}^\nu) \otimes \mathcal{O}_Y(-(\nu-1)\cdot H) = \Big(\bigotimes^r f_*\omega_{X/Y}^\nu\Big) \otimes \mathcal{O}_Y(-(\nu-1)\cdot H) \\ &\qquad\qquad \longrightarrow S^r(f_*\omega_{X/Y}^\nu) \otimes \mathcal{O}_Y(-(\nu-1)\cdot H). \end{aligned}$$
Since the quotient of a weakly positive sheaf is weakly positive, the sheaf on the right hand side is weakly positive over U_0, hence $f_*\omega_{X/Y}^\nu$ is ample with respect to U_0. For ν sufficiently large $\mathcal{O}_Y((\nu-1)\cdot H - S)$ is ample, and one obtains the second part of 3.4. □

If $f : X \to Y$ is not semi-stable in codimension one, the sheaf of relative n-forms $\Omega_{X/Y}^n(\log\Delta)$ might be strictly smaller than the relative dualizing sheaf $\omega_{X/Y}$. In fact, comparing the first Chern classes of the entries in the tautological sequence

(3.5.5) $$0 \longrightarrow f^*\Omega_Y^1(\log S) \longrightarrow \Omega_X^1(\log\Delta) \longrightarrow \Omega_{X/Y}^1(\log\Delta) \longrightarrow 0$$

one finds for $\Delta = f^*S$

(3.5.6) $$\Omega_{X/Y}^n(\log\Delta) = \omega_{X/Y}(\Delta_{\mathrm{red}} - \Delta).$$

Corollary 3.6. *Under the assumptions made in 3.4, i), ii) or iii), for all ν sufficiently large and divisible, the sheaf $f_*\Omega_{X/Y}^n(\log\Delta)^\nu$ is ample with respect to U_0.*

Before proving 3.6 let us start to study the behavior of the relative q-forms under base extensions. Here we will prove a more general result than needed for 3.6, and we will not require $Y \setminus U$ to be smooth.

Assumptions 3.7. Let $f : X \to Y$ be any partial compactification of a smooth family $V \to U$, let $\psi : Y' \to Y$ be a finite covering with Y' non singular, and let $\tilde{X}$ be the normalization of $X \times_Y Y'$. Consider a desingularization $\varphi : X' \to \tilde{X}$, where we assume the center of φ to lie in the singular locus of $\tilde{X}$. The induced morphisms are denoted by

(3.7.1)
$$\begin{tikzcd}
X' \arrow[r, "\varphi"] \arrow[dr, "f'"'] & \tilde{X} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\tilde{f}"] & X \times_Y Y' \arrow[r, "\pi_1"] \arrow[dl, "\pi_2"] & X \arrow[dl, "f"] \\
 & Y' \arrow[r, "\psi"] & Y. &
\end{tikzcd}$$

Let us define $\delta = \tilde{\varphi} \circ \varphi : X' \to X \times_Y Y'$ and $\psi' = \pi_1 \circ \delta : X' \to X$. Finally we write $S' = \psi^* S$ and $\Delta' = \psi'^* \Delta$. The discriminant loci of ψ and ψ' will be $\Delta(X'/X)$, and $\Delta(Y'/Y)$, respectively. We will assume that $S + \Delta(Y'/Y)$ and $\Delta + \Delta(X'/X)$, as well as their preimages in Y' and X', are normal crossing divisors.

Lemma 3.8. *Using the assumptions and notations from 3.7,*

i) *there exists for all p an injection*

$$\psi'^* \Omega^p_{X/Y}(\log \Delta) \xrightarrow{\subset} \Omega^p_{X'/Y'}(\log \Delta'),$$

which is an isomorphism over $\psi'^{-1}(X \setminus \mathrm{Sing}(\Delta))$.

ii) *there exists for all $\nu > 0$ an injection*

$$f'_*(\Omega^n_{X'/Y'}(\log \Delta')^{(\nu-1)} \otimes \omega_{X'/Y'}) \longrightarrow \psi^* f_*(\Omega^n_{X/Y}(\log \Delta)^{(\nu-1)} \otimes \omega_{X/Y}),$$

which is an isomorphism over $\psi^{-1}(U)$.

Proof. If one replaces in the tautological sequence (3.5.5) the divisor S by a larger one, the sheaf on the right hand side does not change, hence

$$\Omega^1_{X/Y}(\log \Delta) = \Omega^1_{X/Y}(\log(\Delta + \Delta(X'/X))).$$

Both, $\Omega^1_Y(\log(S + \Delta(Y'/Y)))$ and $\Omega^1_X(\log(\Delta + \Delta(X'/X)))$ behave well under pullback to X' (see [6], 3.20, for example). To be more precise, there exists an isomorphism

$$\psi^* \Omega^1_Y(\log(S + \Delta(Y'/Y))) \simeq \Omega^1_{Y'}(\log(S' + \psi^* \Delta(Y'/Y)))$$

and an injection

$$\psi'^* \Omega^1_X(\log(\Delta + \Delta(X'/X))) \xrightarrow{\subset} \Omega^1_{X'}(\log \psi'^*(\Delta + \Delta(X'/X)))$$

which is an isomorphism over the largest open subscheme V_1', where ψ' is an isomorphism. Since $\tilde{X}$ is non-singular outside of Δ, and since the singularities of $\tilde{X}$ can only appear over singular points of the discriminant $\Delta(X'/X)$, we find $\psi'^{-1}(X \setminus \mathrm{Sing}(\Delta)) \subset V_1'$.

For ii) we use again that $\tilde{X}$ is non-singular over $X \setminus \mathrm{Sing}(\Delta)$. So part i) induces an isomorphism

$$\varphi_*(\Omega^n_{X'/Y'}(\log \Delta')^{(\nu-1)} \otimes \omega_{X'/Y'}) \xrightarrow{\simeq} \tilde{\varphi}^* \pi_1^* (\Omega^n_{X/Y}(\log \Delta)^{(\nu-1)}) \otimes \omega_{\tilde{X}/Y'}.$$

The natural map $\tilde{\varphi}_* \omega_{\tilde{X}/Y'} \longrightarrow \omega_{X \times_Y Y'/Y'}$ and the projection formula give

$$\delta_*(\Omega^n_{X'/Y'}(\log \Delta')^{(\nu-1)} \otimes \omega_{X'/Y'}) \longrightarrow \pi_1^*(\Omega^n_{X/Y}(\log \Delta)^{(\nu-1)} \otimes \omega_{X/Y}),$$

and ii) follows by flat base change. □

Proof of 3.6. Applying 2.3, b), one finds a finite covering $\psi : Y' \to Y$ such that the family $f' : X' \to Y'$ is semi-stable in codimension one, hence (3.5.6) implies $\omega_{X'/Y'} = \Omega^n_{X'/Y'}(\log \Delta')$, whereas $\omega_{X/Y} \otimes f^* \mathcal{O}_Y(-S) \subset \Omega^n_{X/Y}(\log(\Delta))$. So 3.8, ii), gives a morphism of sheaves

$$\begin{aligned} f'_*(\omega^\nu_{X'/Y'}) \otimes \mathcal{O}_{Y'}(-\psi^* S) &\longrightarrow \psi^*(f_*(\Omega^n_{X/Y}(\log \Delta)^{(\nu-1)} \otimes \omega_{X/Y}) \otimes \mathcal{O}_Y(-S)) \\ &\longrightarrow \psi^* f_*(\Omega^n_{X/Y}(\log \Delta)^\nu) \end{aligned}$$

By 3.4, iii), for some $\nu \gg 0$ the sheaf on the left hand side will be ample with respect to $\psi^{-1}(U_0)$, hence the sheaf on the right hand side has the same property. □

A positivity property, similar to the last one, will be expressed in terms of fibred products of the given family. It will be used in the proof of 1.4, iii). We do not need it in its full strength, just "up to codimension two in Y". Nevertheless, in order to be able to refer to [25], we formulate it in a more general setup.

Let $V \to U$ be a smooth family with $\omega_{V/U}$ f-semi-ample. By 2.6 we find a smooth birational model $\tilde{V} \to \tilde{U}$ whose compactification $f : X \to Y$ fits into the diagram (2.6.1):

$$\begin{tikzcd} X & X' \arrow[l, "\psi'"'] \arrow[d, "f'"] & Z \arrow[l, "\sigma"'] \arrow[dl, "g"] \\ Y & Y' \arrow[l, "\psi"'] & \end{tikzcd}$$

with $X \xrightarrow{f} Y$ the left vertical map.

Let us choose any $\nu \geq 3$ such that

$$f^* f_* \omega^\nu_{X/Y} \longrightarrow \omega^\nu_{X/Y}$$

is surjective over $\tilde{V}$, and that the multiplication map

$$S^\eta(f_* \omega^\nu_{X/Y}) \longrightarrow f_* \omega^{\eta \cdot \nu}_{X/Y}$$

is surjective over $\tilde{U}$. By definition one has $\mathrm{Var}(f) = \mathrm{Var}(g)$. If $\mathrm{Var}(f) = \dim(Y)$, applying 3.4, i), to g one finds that the sheaf λ_ν, defined in 2.6, d), is of maximal Kodaira dimension. Hence some power of λ_ν is of the form $\mathcal{A}(D)$, for an ample invertible sheaf $\mathcal{A}$ on Y and for an effective divisor D on Y. We may assume moreover, that $D \geq S$ and, replacing the number N_ν in 2.6 by some multiple, that

$$\det(g_* \omega^\nu_{Z/Y'})^{N_\nu} = \mathcal{A}(D)^{\nu \cdot (\nu-1) \cdot e}$$

where $e = \mathrm{Max}\{e(\omega^\nu_F);\ F \text{ a fibre of } V \to U\}$.

Proposition 3.9. *For $r = N_\nu \cdot \mathrm{rank}(f_*\omega^\nu_{X/Y})$, let $X^{(r)}$ denote a desingularization of the r-th fibre product $X \times_Y \ldots \times_Y X$ and let $f^{(r)} : X^{(r)} \to Y$ be the induced family. Then for all β sufficiently large and divisible the sheaf*

$$f^{(r)}_*(\Omega^{r\cdot n}_{X^{(r)}/Y}(\log \Delta)^{\beta\cdot\nu}) \otimes \mathcal{A}^{-\beta\cdot\nu\cdot(\nu-2)}$$

is globally generated over some non-empty open subset U_0 of $\tilde{U}$, and the sheaf

$$\Omega^{r\cdot n}_{X^{(r)}/Y}(\log \Delta)^{\beta\cdot\nu} \otimes f^{(r)*}\mathcal{A}^{-\beta\cdot\nu\cdot(\nu-2)}$$

is globally generated over ${f^{(r)}}^{-1}(U_0)$.

Proof. By [25], 4.1, the sheaf

$$f^{(r)}_*(\omega^{\beta\cdot\nu}_{X^{(r)}/Y}) \otimes \mathcal{A}^{-\beta\cdot\nu\cdot(\nu-2)} \otimes \mathcal{O}_Y(-\beta\cdot\nu\cdot(\nu-1)\cdot D)$$

is globally generated over some open subset. However, by (3.5.6)

$$\omega^{\beta\cdot\nu}_{X^{(r)}/Y} \otimes f^*\mathcal{O}_Y(-\beta\cdot\nu\cdot S)$$

is contained in

$$\Omega^{r\cdot n}_{X^{(r)}/Y}(\log \Delta)^{\beta\cdot\nu}.$$

Since

$$\beta\cdot\nu\cdot(\nu-1)\cdot D \geq \beta\cdot\nu\cdot S$$

one obtains 3.9, as stated. □

4. Higgs bundles and the proof of 1.4

As in [24] and [25], in order to prove 1.4 we have to construct certain Higgs bundles, and we have to compare them to one, induced by a variation of Hodge structures. For 1.4, iii), we will just use the content of the second half of section 3. For iv) we need in addition Kawamata's covering construction, as explained in 2.3. The reduction steps contained in the second half of section 2 will be needed for 1.4, i).

So let U be a manifold and let Y be a smooth projective compactification with $Y \setminus U$ a normal crossing divisor. Starting with a smooth family $V \to U$ with $\omega_{V/U}$ relative semi-ample over U, we first choose a smooth projective compactification X of V, such that $V \to U$ extends to $f : X \to Y$.

In the first half of the section, we will work with good partial compactifications as defined in 2.1. Hence leaving out a codimension two subscheme of Y, we will assume that the divisor $S = Y \setminus U$ is smooth, that f is flat and that $\Delta = X \setminus V$ is a relative normal crossing divisor. The exact sequence (3.5.5) induces a filtration on the wedge product $\Omega^p_{X/Y}(\log \Delta)$, and thereby the tautological sequences

(4.0.1)
$$0 \to f^*\Omega^1_Y(\log S) \otimes \Omega^{p-1}_{X/Y}(\log \Delta) \to \mathfrak{gr}(\Omega^p_X(\log \Delta)) \to \Omega^p_{X/Y}(\log \Delta) \to 0,$$

where

$$\mathfrak{gr}(\Omega^p_X(\log \Delta)) = \Omega^p_X(\log \Delta)/f^*\Omega^2_Y(\log S) \otimes \Omega^{p-2}_{X/Y}(\log \Delta).$$

Given an invertible sheaf $\mathcal{L}$ on X we will study in this section various sheaves of the form

$$F_0^{p,q} := R^q f_*(\Omega^p_{X/Y}(\log \Delta) \otimes \mathcal{L}^{-1})/_{\text{torsion}}$$

together with the edge morphisms

$$\tau^0_{p,q} : F_0^{p,q} \longrightarrow F_0^{p-1,q+1} \otimes \Omega^1_Y(\log S),$$

induced by the exact sequence (4.0.1), tensored with $\mathcal{L}^{-1}$.

First we have to extend the base change properties for direct images, studied in 3.8, ii), to higher direct images.

Lemma 4.1. *Keeping the notations and assumptions from 3.7, let Y'_1 be the largest open subset in Y' with $X \times_Y Y'_1$ normal. We write*

$$\iota : \psi^*\Omega^1_Y(\log S) \longrightarrow \Omega^1_{Y'}(\log S')$$

for natural inclusion, and we consider an invertible sheaf $\mathcal{L}$ on X, and its pullback $\mathcal{L}' = \psi'^*\mathcal{L}$ to X'.

Then for all p and q, there are morphisms

$$\psi^* F_0^{p,q} \xrightarrow{\zeta_{p,q}} F_0'^{p,q} := R^q f'_*(\Omega^p_{X'/Y'}(\log \Delta') \otimes \mathcal{L}'^{-1})/_{\text{torsion}},$$

whose restriction to Y'_1 are isomorphisms, and for which the diagram

$$\begin{array}{ccc} \psi^* F_0^{p,q} & \xrightarrow{\psi^*(\tau^0_{p,q})} & \psi^* F_0^{p-1,q+1} \otimes \Omega^1_Y(\log S)) \\ \downarrow{\scriptstyle \zeta_{p,q}} & & \downarrow{\scriptstyle \zeta_{p-1,q+1}\otimes\iota} \\ F_0'^{p,q} & \xrightarrow{\tau'^0_{p,q}} & F_0'^{p-1,q+1} \otimes \Omega^1_{Y'}(\log S') \end{array} \tag{4.1.1}$$

commutes. Here ${\tau'}^0_{p,q}$ is again the edge morphism induced by the exact sequence on X', corresponding to (4.0.1) and tensored with $\mathcal{L}'^{-1}$.

Proof. We use the notations from (3.7.1), i.e.

$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[dr, "f'"'] & \tilde{X} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\tilde{f}"] & X \times_Y Y' \arrow[r, "\pi_1"] \arrow[dl, "\pi_2"] & X \arrow[dl, "f"] \\ & Y' \arrow[r, "\psi"] & Y & \end{tikzcd}$$

and $\psi' = \varphi \circ \tilde{\varphi} \circ \pi_1$. As in the proof of 3.8, in order to show the existence of the morphisms $\zeta_{p,q}$ and the commutativity of the diagram (4.1.1) we may enlarge S and S' to include the discriminant loci, hence assume that

$$\psi^*\Omega^1_Y(\log S) = \Omega^1_{Y'}(\log S').$$

By the generalized Hurwitz formula [6], 3.21,

$$\psi'^*\Omega^p_X(\log \Delta) \subset \Omega^p_{X'}(\log \Delta'),$$

and by [5], Lemme 1.2,

$$R^q\varphi_*\Omega^p_{X'}(\log \Delta') = \begin{cases} \tilde{\varphi}^*\pi_1^*\Omega^p_X(\log \Delta) & \text{for } q = 0 \\ 0 & \text{for } q > 0. \end{cases}$$

The tautological sequence

$$0 \to f'^*\Omega^1_{Y'}(\log(S')) \to \Omega^1_{X'}(\log\Delta') \to \Omega^1_{X'/Y'}(\log\Delta') \to 0$$

defines a filtration on $\Omega^p_{X'}(\log\Delta')$, with subsequent quotients isomorphic to

$$f'^*\Omega^\ell_{Y'}(\log S')\otimes\Omega^{p-\ell}_{X'/Y'}(\log\Delta').$$

Induction on p allows to deduce that

$$R^q\varphi_*\Omega^p_{X'/Y'}(\log\Delta') = \begin{cases} \tilde{\varphi}^*\pi_1^*\Omega^p_{X/Y}(\log\Delta) & \text{for } q=0 \\ 0 & \text{for } q>0. \end{cases} \tag{4.1.2}$$

On the other hand, the inclusion $\mathcal{O}_{Z\times_Y Y'}\to\tilde{\varphi}_*\mathcal{O}_{\tilde{Z}}$ and flat base change gives

$$\psi^*F_0^{p,q} = \psi^*R^qf_*(\Omega^p_{X/Y}(\log\Delta)\otimes\mathcal{L}^{-1}) \xrightarrow{\simeq} \tag{4.1.3}$$

$$R^q\pi_{2*}(\pi_1^*(\Omega^p_{X/Y}(\log\Delta)\otimes\mathcal{L}^{-1}))\to R^q\tilde{f}_*(\tilde{\varphi}^*(\pi_1^*(\Omega^p_{X/Y}(\log\Delta)\otimes\mathcal{L}^{-1}))) = F_0'^{p,q},$$

hence $\zeta_{p,q}$. The second morphism in (4.1.3) is an isomorphism on the largest open subset where $\tilde{\varphi}$ is an isomorphism, in particular on $\tilde{f}^{-1}(Y_1')$.

The way we obtained (4.1.2) the morphisms are obviously compatible with the different tautological sequences. Since we assumed S to contain the discriminant locus, the pull back of (4.0.1) to $\tilde{X}$ is isomorphic to

$$0\to\varphi_*(f'^*\Omega^1_{Y'}(\log S')\otimes\Omega^{p-1}_{X'/Y'}(\log\Delta'))\to\varphi_*(\mathfrak{gr}(\Omega^p_{X'}(\log\Delta')))$$
$$\to\varphi_*(\Omega^p_{X'/Y'}(\log\Delta'))\to 0,$$

and the diagram (4.1.1) commutes. □

Remark 4.2. If $\psi : Y'\to Y$ is any smooth morphism, then again $X\times_Y Y'$ non-singular. The compatibility of the $F_0^{p,q}$ with pullback, i.e. the existence of an isomorphism $\zeta_{p,q} : \psi^*F_0^{p,q}\to F_0'^{p,q}$, and the commutativity of (4.1.1) is also guaranteed, in this case. In fact, both φ and $\tilde{\varphi}$ are isomorphisms, as well as the two morphisms in (4.1.3).

Corollary 4.3. *Keeping the assumptions from 4.1, assume that $X\times_Y Y'$ is normal. Then the image of*

$$F_0'^{p,q}\xrightarrow{\tau'^0_{p,q}} F_0'^{p-1,q+1}\otimes\Omega^1_{Y'}(\log S')$$

lies in $F_0'^{p-1,q+1}\otimes\psi^*(\Omega^1_Y(\log S))$.

□

In the sequel we will choose $\mathcal{L}=\Omega^n_{X/Y}(\log\Delta)$. Let us consider first the case that for some $\nu\gg 1$ and for some invertible sheaf $\mathcal{A}$ on Y the sheaf

$$\mathcal{L}^\nu\otimes f^*\mathcal{A}^{-\nu} \quad \text{is globally generated over} \quad V_0=f^{-1}(U_0), \tag{4.3.1}$$

for some open dense subset U_0 of Y.

We will recall some of the constructions performed in [25], §6. Let H denote the zero divisor of a section of $\mathcal{L}^\nu\otimes f^*\mathcal{A}^{-\nu}$, whose restriction to a general fibre of f is non-singular. Let T denote the closure of the discriminant of $H\cap V\to U$. Leaving out some more codimension two subschemes, we may

assume that $S+T$ is a smooth divisor. We will write $\Sigma = f^*T$ and we keep the notation $\Delta = f^*(S)$.

Let $\delta : W \to X$ be a blowing up of X with centers in $\Delta + \Sigma$ such that $\delta^*(H + \Delta + \Sigma)$ is a normal crossing divisor. We write

$$\mathcal{M} = \delta^*(\Omega^n_{X/Y}(\log \Delta) \otimes f^*\mathcal{A}^{-1}).$$

Then for $B = \delta^* H$ one has $\mathcal{M}^\nu = \mathcal{O}_W(B)$. As in [6], §3, one obtains a cyclic covering of W, by taking the ν-th root out of B. We choose Z to be a desingularization of this covering and we denote the induced morphisms by $g : Z \to Y$, and $h : W \to Y$. Writing $\Pi = g^{-1}(S \cup T)$, the restriction of g to $Z_0 = Z \setminus \Pi$ will be smooth.

For the normal crossing divisor B we define

$$\mathcal{M}^{(-1)} = \mathcal{M}^{-1} \otimes \mathcal{O}_W\Big(\Big[\frac{B}{\nu}\Big]\Big), \quad \text{and} \quad \mathcal{L}^{(-1)} = \delta^*(\mathcal{L}^{-1}) \otimes \mathcal{O}_W\Big(\Big[\frac{B}{\nu}\Big]\Big).$$

In particular the cokernel of the inclusion $\delta^*\mathcal{L}^{-1} \subset \mathcal{L}^{(-1)}$ lies in $h^{-1}(S+T)$. The sheaf

$$\begin{aligned} F^{p,q} &= R^q h_*(\delta^*(\Omega^p_{X/Y}(\log \Delta)) \otimes \mathcal{M}^{(-1)}) \otimes \mathcal{A}^{-1}/_{\text{torsion}} \\ &= R^q h_*(\delta^*(\Omega^p_{X/Y}(\log \Delta)) \otimes \mathcal{L}^{(-1)})/_{\text{torsion}} \end{aligned}$$

contains the sheaf $F_0^{p,q}$ and both are isomorphic outside of $S+T$. The edge morphism

$$\tau_{p,q} : F^{p,q} \longrightarrow F^{p-1,q+1} \otimes \Omega^1_Y(\log S)$$

given by the tautological exact sequence

$$\begin{aligned} 0 \to h^*\Omega^1_Y(\log S) \otimes \delta^*(\Omega^{p-1}_{X/Y}(\log \Delta)) \otimes \mathcal{L}^{(-1)} \to \\ \delta^*(\mathfrak{gr}(\Omega^p_X(\log \Delta))) \otimes \mathcal{L}^{(-1)} \to \delta^*(\Omega^p_{X/Y}(\log \Delta)) \otimes \mathcal{L}^{(-1)} \to 0 \end{aligned}$$

is compatible with $\tau^0_{p,q}$. Let us remark, that the sheaves $F^{p,q}$ depend on the choice of the divisor H and they can only be defined assuming (4.3.1).

Up to now, we constructed two Higgs bundles

$$F_0 = \bigoplus F_0^{p,q} \xrightarrow{\subset} F = \bigoplus F^{p,q}.$$

We will see below, that $\mathcal{A} \otimes F$ can be compared with a Higgs bundle E, given by a variation of Hodge structures. This will allow to use the negativity of the kernel of Kodaira-Spencer maps (see [26]), to show that $\mathrm{Ker}(\tau_{p,q})^\vee$ is big.

By [4], for all $k \geq 0$, the local constant system $R^k g_* \mathbb{C}_{Z_0}$ gives rise to a local free sheaf $\mathcal{V}_k$ on Y with the Gauß-Manin connection

$$\nabla : \mathcal{V}_k \longrightarrow \mathcal{V}_k \otimes \Omega^1_Y(\log(S+T)).$$

We assume that $\mathcal{V}_k$ is the quasi-canonical extension of

$$(R^k g_* \mathbb{C}_{Z_0}) \otimes_{\mathbb{C}} \mathcal{O}_{Y \setminus (S \cup T)},$$

i.e. that the real part of the eigenvalues of the residues around the components of $S+T$ lie in $[0,1)$.

Since we assumed $S+T$ to be non-singular, $\mathcal{V}_k$ carries a filtration $\mathcal{F}^p$ by subbundles (see [18]). So the induced graded sheaves $E^{p,k-p}$ are locally free,

and they carry a Higgs structure with logarithmic poles along $S+T$. Let us denote it by

$$(\mathfrak{gr}_{\mathcal{F}}(\mathcal{V}_k), \mathfrak{gr}_{\mathcal{F}}(\nabla)) = (E,\theta) = \Big(\bigoplus_{q=0}^{k} E^{k-q,q} \ , \ \bigoplus_{q=0}^{k} \theta_{k-q,q} \Big).$$

As well-known (see for example [7], page 130) the bundles $E^{p,q}$ are given by

$$E^{p,q} = R^q g_* \Omega^p_{Z/Y}(\log \Pi).$$

Writing again $\mathfrak{gr}(\_)$ for "modulo the pullback of 2-forms on Y", the Gauß-Manin connection is the edge morphism of

$$0 \to g^*\Omega^1_Y(\log(S+T)) \otimes \Omega^{\bullet-1}_{Z/Y}(\log \Pi) \to \mathfrak{gr}(\Omega^\bullet_Z(\log \Pi)) \to \Omega^\bullet_{Z/Y}(\log \Pi) \to 0.$$

Hence the Higgs maps

$$\theta_{p,q} : E^{p,q} \longrightarrow E^{p-1,q+1} \otimes \Omega^1_Y(\log(S+T))$$

are the edge morphisms of the tautological exact sequences

$$0 \to g^*\Omega^1_Y(\log(S+T)) \otimes \Omega^{p-1}_{Z/Y}(\log \Pi) \to \\ \mathfrak{gr}(\Omega^p_Z(\log \Pi)) \to \Omega^p_{Z/Y}(\log \Pi) \to 0.$$

In the sequel we will write $T_*(-\log **)$ for the dual of $\Omega^1_*(\log **)$.

Lemma 4.4. *Under the assumption (4.3.1) and using the notations introduced above, let*

$$\iota : \Omega^1_Y(\log S) \longrightarrow \Omega^1_Y(\log(S+T))$$

be the natural inclusion. Then there exist morphisms $\rho_{p,q} : \mathcal{A} \otimes F^{p,q} \to E^{p,q}$ *such that:*

i) *The diagram*

$$\begin{array}{ccc} E^{p,q} & \xrightarrow{\theta_{p,q}} & E^{p-1,q+1} \otimes \Omega^1_Y(\log(S+T)) \\ {\scriptstyle\rho_{p,q}}\uparrow & & \uparrow{\scriptstyle\rho_{p-1,q+1}\otimes\iota} \\ \mathcal{A} \otimes F^{p,q} & \xrightarrow{\mathrm{id}_{\mathcal{A}} \otimes \tau_{p,q}} & \mathcal{A} \otimes F^{p-1,q+1} \otimes \Omega^1_Y(\log S). \end{array}$$

commutes.

ii) $F^{n,0}$ *has a section* $\mathcal{O}_Y \to F^{n,0}$, *which is an isomorphism on* $Y \setminus (S \cup T)$.

iii) $\tau_{n,0}$ *induces a morphism*

$$\tau^\vee : T_Y(-\log S) = (\Omega^1_Y(\log S))^\vee \longrightarrow F^{n,0^\vee} \otimes F^{n-1,1},$$

which coincides over $Y \setminus (S \cup T)$ *with the Kodaira-Spencer map*

$$T_Y(-\log S) \longrightarrow R^1 f_* T_{X/Y}(-\log \Delta).$$

iv) $\rho_{n,0}$ *is injective. If the general fibre of* f *is canonically polarized, then the morphisms* $\rho_{n-m,m}$ *are injective, for all* m.

v) *Let* $\mathcal{K}^{p,q} = \mathrm{Ker}(E^{p,q} \xrightarrow{\theta_{p,q}} E^{p-1,q+1} \otimes \Omega^1_Y(\log(S+T)))$. *Then the dual* $(\mathcal{K}^{p,q})^\vee$ *is weakly positive with respect to some open dense subset of* Y.

vi) *The composite*

$$\theta_{n-q+1,q-1} \circ \cdots \circ \theta_{n,0} : E^{n,0} \longrightarrow E^{n-q,q} \otimes \bigotimes^{q} \Omega^1_Y(\log(S+T))$$

factors like

$$E^{n,0} \xrightarrow{\theta^q} E^{n-q,q} \otimes S^q\Omega^1_Y(\log(S+T)) \xrightarrow{\subset} E^{n-q,q} \otimes \bigotimes^{q} \Omega^1_Y(\log(S+T)).$$

Proof. The properties i) - iv) have been verified in [25], 6.3 in case the general fibre is canonically polarized. So let us just sketch the arguments.

By [6] (see also [25], 6.2) the sheaf

$$R^q h_*(\Omega^p_{W/Y}(\log(B+\delta^*\Delta+\delta^*\Sigma)) \otimes \mathcal{M}^{(-1)})$$

is a direct factor of $E^{p,q}$. The morphism $\rho_{p,q}$ is induced by the natural inclusions

$$\begin{aligned} &\delta^*\Omega^p_{X/Y}(\log\Delta) \to \delta^*\Omega^p_{X/Y}(\log(\Delta+\Sigma)) \\ &\qquad \to \Omega^p_{W/Y}(\log(\delta^*\Delta+\delta^*\Sigma)) \to \Omega^p_{W/Y}(\log(B+\delta^*\Delta+\delta^*\Sigma)), \end{aligned}$$

tensored with $\mathcal{M}^{(-1)} = \mathcal{L}^{(-1)} \otimes h^*\mathcal{A}$.

Such an injection also exist for Y replaced by $\mathrm{Spec}(\mathbb{C})$. Since the different tautological sequences are compatible with those inclusions one obtains i). Over $Y \setminus (S \cup T)$ the kernel of $\rho_{n-m,m}$ is a quotient of the sheaf

$$R^{m-1}(h|_B)_*(\Omega^{n-m-1}_{B/Y} \otimes \mathcal{M}^{-1}|_B).$$

In particular $\rho_{n,0}$ is injective. The same holds true for all the $\rho_{n-m,m}$ in case $\mathcal{M}$ is fibre wise ample, by the Akizuki-Kodaira-Nakano vanishing theorem.

By definition

$$F^{n,0} = h_*(\delta^*(\Omega^n_{X/Y}(\log\Delta)) \otimes \mathcal{L}^{(-1)}) = h_*\mathcal{O}_W\Big(\Big[\frac{B}{\nu}\Big]\Big),$$

and ii) holds true.

For iii), recall that over $Y \setminus (S \cup T)$ the morphism

$$\delta^*(\mathcal{L} \otimes f^*\mathcal{A}^{-1}) = \mathcal{M}^{-1} \to \mathcal{M}^{(-1)}$$

is an isomorphism. By the projection formula the morphism $\tau_{n,0}|_{Y\setminus(S\cup T)}$ is the restriction of the edge morphism of the short exact sequence

$$0 \to f^*\Omega^1_U \otimes \Omega^{n-1}_{V/U} \otimes \mathcal{L}^{-1} \to \mathfrak{gr}(\Omega^n_V) \otimes \mathcal{L}^{-1} \to \Omega^n_{V/U} \otimes \mathcal{L}^{-1} \to 0.$$

The sheaf on the right hand side is $\mathcal{O}_V$ and the one on the left hand side is $f^*\Omega^1_U \otimes T_{V/U}$. For $r = \dim(U)$, tensoring the exact sequence with

$$f^*T_U = f^*(\Omega^{r-1}_U \otimes \omega^{-1}_U)$$

and dividing by the kernel of the wedge product

$$f^*\Omega^1_U \otimes f^*(\Omega^{r-1}_U \otimes \omega^{-1}_U) = f^*\Omega^1_U \otimes f^*T_U \longrightarrow \mathcal{O}_V$$

on the left hand side, one obtains an exact sequence

$$0 \longrightarrow T_{V/U} \longrightarrow \mathcal{G} \longrightarrow f^*T_U \longrightarrow 0, \tag{4.4.1}$$

where $\mathcal{G}$ is a quotient of $\mathfrak{gr}(\Omega^n_V)\otimes{\omega_V}^{-1}\otimes f^*\Omega^{r-1}_U$. By definition, the restriction to $Y\setminus(S\cup T)$ of the morphism considered in iii) is the first edge morphism in the long exact sequence, obtained by applying $R^\bullet f_*$ to (4.4.1).

The wedge product induces a morphism

$$\Omega^n_V\otimes{\omega_V}^{-1}\otimes f^*\Omega^{r-1}_U\longrightarrow\Omega^{n+r-1}_V\otimes\omega_V^{-1}=T_V.$$

This morphism factors through $\mathcal{G}$. Hence the exact sequence (4.4.1) is isomorphic to the tautological sequence

$$0\longrightarrow T_{V/U}\longrightarrow T_V\longrightarrow f^*T_U\longrightarrow 0. \tag{4.4.2}$$

The edge morphism $T_U\to R^1f_*T_{V/U}$ of (4.4.2) is the Kodaira-Spencer map.

In order to prove v), we use as in the proof of 2.3, b), Kawamata's covering construction to find a non-singular finite covering $\rho: Y'\to Y$ such that for some desingularization Z' of $Z\times_Y Y'$ the induced variation of Hodge structures has uni-potent monodromy, and such that $g': Z'\to Y'$ is semi-stable.

From 4.1, applied to Z, $\mathcal{O}_Z$ and Π instead of X, $\mathcal{L}$ and Δ one obtains a commutative diagram

$$\begin{array}{ccc}
\rho^*E^{p,q} & \xrightarrow{\rho^*\theta_{p,q}} & \rho^*E^{p-1,q+1}\otimes\Omega^1_Y(\log S+T)\\
\subset\big\downarrow & & \subset\big\downarrow\\
E'^{p,q} & \xrightarrow{\theta'_{p,q}} & E'^{p-1,q+1}\otimes\Omega^1_{Y'}(\log S'),
\end{array}$$

where $S'=\psi^*(S+T)$, where $E'^{p,q}=R^qg'_*\Omega^p_{Z'/Y'}(\log\Pi')$, and where $\theta'_{p,q}$ is the edge-morphism.

In particular the pullback of the kernel of $\theta_{p,q}$, the sheaf $\rho^*\mathcal{K}^{p,q}$, lies in the kernel $\mathcal{K}'^{p,q}$ of $\theta'_{p,q}$. Leaving out some codimension two subschemes of Y and Y', we may assume that $\mathcal{K}'^{p,q}$ is a subbundle of $E'^{p,q}$. Choose a smooth extension $\bar{Y}'$ of Y' such that the closure of $S'\cup(\bar{Y}'-Y')$ is a normal crossing divisor, and let $\bar{E}'^{p,q}$ be the Higgs bundle, corresponding to the canonical extension of the variation of Hodge structures. For some choice of the compactification $\mathcal{K}'^{p,q}$ will extend to a subbundle $\bar{\mathcal{K}}'^{p,q}$ of $\bar{E}'^{p,q}$. By [26], 1.2, the dual $(\bar{\mathcal{K}}'^{p,q})^\vee$ is numerically effective, hence weakly positive. Thereby $\rho^*(\mathcal{K}^{p,q})^\vee$ is weakly positive over some open subset, and the compatibility of weak positivity with pullback shows v).

For vi) one just has to remark that on page 12 of [19] it is shown that $\theta\wedge\theta=0$ for

$$\theta=\bigoplus_{q=0}^n\theta_{n-q,q}.$$

□

Corollary 4.5. *Assume (4.3.1) holds true for some ample invertible sheaf $\mathcal{A}$ and for some $\nu\gg 1$. Assume moreover that there exists a locally free subsheaf Ω of $\Omega^1_Y(\log S)$ such that $\mathrm{id}_{\mathcal{A}}\otimes\tau_{p,q}$ factors through*

$$\mathcal{A}\otimes F^{n-q,q}\longrightarrow\mathcal{A}\otimes F^{n-q-1,q+1}\otimes\Omega,$$

for all q. Then for some $0<m\le n$ there exists a big coherent subsheaf $\mathcal{P}$ of $S^m(\Omega)$.

Proof. Using the notations from 4.4, write $\mathcal{A}\otimes \tilde{F}^{n-q,q} = \rho_{n-q,q}(\mathcal{A}\otimes F^{n-q,q})$. By 4.4, i), and by the choice of Ω

$$\theta_{n-q,q}(\mathcal{A}\otimes \tilde{F}^{n-q,q}) \subset \mathcal{A}\otimes \tilde{F}^{n-q-1,q+1}\otimes\Omega.$$

By 4.4, ii) and iv), there is a section $\mathcal{O}_Y \to F^{n,0}\simeq \tilde{F}^{n,0}$, generating $\tilde{F}^{n,0}$ over $Y\setminus(S\cup T)$, and by 4.4, v), $\mathcal{A}\otimes\tilde{F}^{n,0}$ can not lie in the kernel of $\theta_{n,0}$. Hence the largest number m with $\theta^m(\mathcal{A}\otimes\tilde{F}^{n,0})\neq 0$ satisfies $1\leq m\leq n$. By the choice of m

$$\theta^{m+1}(\mathcal{A}\otimes\tilde{F}^{n,0})=0,$$

and 4.4, vi) implies that $\theta^m(\mathcal{A}\otimes\tilde{F}^{n,0})$ lies in

$$(\mathcal{K}^{n-m,m}\cap\mathcal{A}\otimes\tilde{F}^{n-m,m})\otimes S^m(\Omega)\subset\mathcal{K}^{n-m,m}\otimes S^m(\Omega).$$

We obtain morphisms of sheaves

$$\mathcal{A}\otimes(\mathcal{K}^{n-m,m})^\vee \xrightarrow{\subset} \mathcal{A}\otimes\tilde{F}^{n,0}\otimes(\mathcal{K}^{n-m,m})^\vee \xrightarrow{\neq 0} S^m(\Omega).$$

By 4.4, v), the sheaf on the left hand side is big, hence its image $\mathcal{P}\subset S^m(\Omega)$ is big as well. □

Proof of 1.4, iii). Let Y be the given smooth projective compactification of U with $Y\setminus U$ a normal crossing divisor. In order to prove iii) we may blow up Y. Hence given a morphism $V\to U$ with $\omega_{V/U}$ semi-ample, by abuse of notations we will assume that $V\to U$ itself fits into the diagram (2.6.1). So we may apply 3.9 and replace X by $X^{(r)}$ for r sufficiently large. In this way we loose control on the dimension of the fibres, but we enforce the existence of a family for which (4.3.1) holds true. We obtain the big coherent subsheaf $\mathcal{P}$, asked for in 1.4, ii), by 4.5, applied to $\Omega=\Omega^1_Y(\log S)$. □

In order to prove 1.4, iv), we have to argue in a slightly different way, since we are not allowed to perform any construction, changing the dimension of the general fibre.

Proof of 1.4, iv). We start again with a smooth projective compactifications X of V, such that $V\to U$ extends to $f: X\to Y$. Recall that for $\mathcal{L}=\Omega^n_{X/Y}(\log\Delta)$, we found in 3.6 some $\nu\gg 1$ and an open dense subset U_0 of Y such that

(4.5.1) $$f_*\mathcal{L}^\nu = f_*\Omega^n_{X/Y}(\log\Delta)^\nu \quad\text{is ample with respect to}\quad U_0$$

(4.5.2) $$\text{and}\quad f^*f_*\mathcal{L}^\nu\longrightarrow\mathcal{L}^\nu\quad\text{is surjective over}\quad V_0=f^{-1}(U_0).$$

Given a very ample sheaf $\mathcal{A}$ on Y, lemma 1.2 implies that for some μ' the sheaf $\mathcal{A}^{-1}\otimes S^{\mu'}(f_*\mathcal{L}^\nu)$ is globally generated over U_0. Lemma 2.3, a), allows to find some smooth covering $\psi: Y'\to Y$ such that $\psi^*\mathcal{A}=\mathcal{A}'^\mu$ for an invertible ample sheaf $\mathcal{A}'$ on Y' and for $\mu=\mu'\cdot\nu$. We will show, that for this covering 1.4, iv), holds true. To this aim, we are allowed to replace Y by the complement of a codimension two subscheme, hence assume that $f: X\to Y$ is a good partial compactification, as defined in 2.1. In particular, we can assume $f_*\mathcal{L}^\nu$ to be locally free. Then the sheaf $\mathcal{L}^\mu\otimes f^*\mathcal{A}^{-1}$ is globally generated over $f^{-1}(U_0)$. Let H be the zero divisor of a general section of this sheaf, and let T denote

the non-smooth locus of $H \to Y$. Leaving out some additional codimension two subset, we may assume that the discriminant $\Delta(Y'/Y)$ does not meet T and the boundary divisor S, hence in particular that the fibred product $X' = X \times_Y Y'$ is smooth. If $\psi' : X' \to X$ and $f' : X' \to Y'$ denote the projections, we write $S' = \psi^* S$, $T' = \psi^* T$, $\Delta' = \psi'^*(\Delta) = h^*(S')$,

$$\mathcal{L}' = \Omega^n_{X'/Y'}(\log \Delta') = \psi'^* \mathcal{L},$$

and so on. The sheaf

$$\mathcal{L}'^{\mu} \otimes f'^* \mathcal{A}'^{-\mu} = \psi'^*(\mathcal{L}^{\mu} \otimes f^* \mathcal{A}^{-1})$$

is globally generated over $\psi^{-1}(V_0)$ and (4.3.1) holds true on Y'. So we can repeat the construction made above, this time over Y' and for the divisor $H' = \psi'^* H$, to obtain the sheaf

$$F'^{p,q} = R^q h'_* \delta'^* (\Omega^p_{X'/Y'}(\log \Delta') \otimes \mathcal{L}'^{(-1)})/_{\text{torsion}},$$

together with the edge morphism

$$\tau'_{p,q} : F'^{p,q} \longrightarrow F'^{p-1,q+1} \otimes \Omega^1_{Y'}(\log S'),$$

induced by the exact sequence

$$0 \to h'^* \Omega^1_{Y'}(\log S') \otimes \delta'^*(\Omega^{p-1}_{X'/Y'}(\log \Delta')) \otimes \mathcal{L}'^{(-1)} \to \\ \delta'^*(\mathfrak{gr}(\Omega^p_{X'}(\log \Delta'))) \otimes \mathcal{L}'^{(-1)} \to \delta'^*(\Omega^p_{X'/Y'}(\log \Delta')) \otimes \mathcal{L}'^{(-1)} \to 0.$$

Returning to the notations from 4.1, the sheaf $F'^{p,q}_0$ defined there is a subsheaf of $F'^{p,q}$, both are isomorphic outside of $S' + T'$ and $\tau'_{p,q}$ commutes with ${\tau'}^0_{p,q}$. By 4.3 the image of $\tau'_{p,q}$ lies in $\psi^*(\Omega^1_Y) \otimes \mathcal{O}_{Y'}(*(S' + T'))$, hence in

$$(\psi^*(\Omega^1_Y) \otimes \mathcal{O}_{Y'}(*(S' + T'))) \cap \Omega_{Y'}(\log S') = \psi^*(\Omega^1_Y(\log S)).$$

By 4.5, for some $1 \leq m \leq n$ the m-th symmetric product of the sheaf

$$\Omega = \psi^*(\Omega^1_Y(\log S))$$

contains a big coherent subsheaf $\mathcal{P}$, as claimed. □

Assume from now on, that the fibres of the smooth family $V \to U$ are canonically polarized, and let $f : X \to Y$ be a partial compactification. The injectivity of $\rho_{n-m,m}$ in 4.4, iv), gives another method to bound the number m in 1.4, iii) and to prove 1.4, ii). For i) we will use in addition, the diagram (2.8.1).

Lemma 4.6. *Using the notations from 4.1, the composite* $\tau^0_{n-q+1,q-1} \circ \cdots \circ \tau^0_{n,0}$ *factors like*

$$F^{n,0}_0 = \mathcal{O}_Y \xrightarrow{\tau^q_0} F^{n-q,q}_0 \otimes S^q(\Omega^1_Y(\log(S))) \xrightarrow{\subset} F^{n-q,q}_0 \otimes \bigotimes^q \Omega^1_Y(\log(S)).$$

Proof. The equality $F^{n,0}_0 = \mathcal{O}_Y$ is obvious by definition. Moreover all the sheaves in 4.6 are torsion free, hence it is sufficient to verify the existence of τ^q_0 on some open dense subset. So we may replace Y by an affine subscheme, and (4.3.1) holds true for $\mathcal{A} = \mathcal{O}_Y$. By 4.4, iv), the sheaves $F^{p,q}_0$ embed in the sheaves $E^{p,q}$, in such a way that $\theta_{p,q}$ restricts to $\tau^0_{p,q}$. One obtains 4.6 from 4.4, vi). □

Proof of 1.4, i) and ii). Replacing Y by the complement of a codimension two subscheme, we may choose a good partial compactification $f : X \to Y$ of $V \to U$. Define

$$\mathcal{N}_0^{p,q} = \mathrm{Ker}(\tau^0_{p,q} : F_0^{p,q} \to F_0^{p-1,q+1} \otimes \Omega^1_Y(\log S)).$$

Claim 4.7. Assume (4.3.1) to hold true for some invertible sheaf $\mathcal{A}$, and let $(\mathcal{N}_0^{p,q})^\vee$ be the dual of the sheaf $\mathcal{N}_0^{p,q}$. Then $\mathcal{A}^{-1} \otimes (\mathcal{N}_0^{p,q})^\vee$ is weakly positive.

Proof. Recall that under the assumption (4.3.1) we have considered above the slightly different sheaf

$$F^{p,q} = R^q h_*(\delta^*(\Omega^p_{X/Y}(\log \Delta)) \otimes \mathcal{L}^{(-1)})/_{\text{torsion}},$$

for $\delta^*\mathcal{L}^{-1} \subset \mathcal{L}^{(-1)}$. So $F_0^{p,q}$ is a subsheaf of $F^{p,q}$ of full rank. The compatibility of $\tau^0_{p,q}$ and $\tau_{p,q}$ implies that $\mathcal{N}_0^{p,q}$ is a subsheaf of

$$\mathcal{N}^{p,q} = \mathrm{Ker}(\tau_{p,q} : F^{p,q} \longrightarrow F^{p-1,q+1} \otimes \Omega^1_Y(\log S)).$$

of maximal rank. Hence the induced morphism

$$(\mathcal{N}^{p,q})^\vee \to (\mathcal{N}_0^{p,q})^\vee$$

is an isomorphism over some dense open subset. By 4.4, iv), the sheaf $\mathcal{A} \otimes F^{p,q}$ is a subsheaf of $E^{p,q}$ and by 4.4, i), the restriction $\theta_{p,q}|_{F^{p,q}}$ coincides with $\mathrm{id}_{\mathcal{A}} \otimes \tau_{p,q}$. Using the notations from 4.4, v), one obtains

$$\mathcal{A} \otimes \mathcal{N}^{p,q} = \mathcal{A} \otimes F^{p,q} \cap \mathcal{K}^{p,q} \xrightarrow{\subset} \mathcal{K}^{p,q}. \tag{4.7.1}$$

By 4.4, v), the dual sheaf $(\mathcal{K}^{p,q})^\vee$ is weakly positive. (4.7.1) induces morphisms

$$(\mathcal{K}^{p,q})^\vee \longrightarrow \mathcal{A}^{-1} \otimes (\mathcal{N}^{p,q})^\vee \longrightarrow \mathcal{A}^{-1} \otimes (\mathcal{N}_0^{p,q})^\vee,$$

surjective over some dense open subset, and we obtain 4.7. □

Claim 4.8.

i) If $\mathrm{Var}(f) = \dim(Y)$, then $(\mathcal{N}_0^{p,q})^\vee$ is big.

ii) In general, for some $\alpha > 0$ and for some invertible sheaf λ of Kodaira dimension $\kappa(\lambda) \geq \mathrm{Var}(f)$ the sheaf

$$S^\alpha((\mathcal{N}_0^{p,q})^\vee) \otimes \lambda^{-1}$$

is generically generated.

Proof. Let us consider as in 3.7 and 4.1 some finite morphism $\psi : Y' \to Y$, a desingularization X' of $X \times_Y Y'$, the induced morphisms $\psi' : X' \to X$ and $f' : X' \to Y'$, $S' = \psi^{-1}(S)$, and $\Delta' = \psi'^{-1}\Delta$. In 4.1 we constructed an injection

$$\psi^*(F_0^{p,q}) \xrightarrow{\zeta_{p,q}} F'^{p,q}_0 = R^q f'_*(\Omega^p_{X'/Y'}(\log \Delta') \otimes \mathcal{L}'^{-1})/_{\text{torsion}},$$

compatible with the edge morphisms $\tau^0_{p,q}$ and $\tau'^0_{p,q}$. Thereby we obtain an injection

$$\psi^*\mathcal{N}_0^{p,q} \xrightarrow{\zeta'_{p,q}} \mathcal{N}'^{p,q}_0 := \mathrm{Ker}(\tau'^0_{p,q}).$$

In case $X \times_Y Y'$ is non-singular, $\zeta_{p,q}$ and hence $\zeta'_{p,q}$ are isomorphisms.

If $\mathrm{Var}(f) = \dim(Y)$, the conditions (4.5.1) and (4.5.2) hold true. As in the proof of 1.4, iv), there exists a finite covering $\psi : Y' \to Y$ with $X \times_Y Y'$ non-singular, such that (4.3.1) holds true for the pullback family $X' \to Y'$. Since a sheaf is ample with respect to some open set, if and only if it has the property on some finite covering, we obtain the bigness of $(\mathcal{N}_0^{p,q})^\vee$ by applying 4.7 to $\psi^*(\mathcal{N}_0^{p,q})^\vee = (\mathcal{N}'^{p,q}_0)^\vee$.

In general 2.2 allows to assume that $X \to Y$ fits into the diagram (2.8.1) constructed in 2.8. Let us write $F_0^{\#p,q}$, and $\mathcal{N}_0^{\#p,q}$ for the sheaves corresponding to $F_0^{p,q}$ and $\mathcal{N}_0^{\#p,q}$ on $Y^\#$ instead of Y.

As we have seen in 4.2 the smoothness of η implies that $\eta^* F^{\#p,q} = F'^{p,q}$, and

$$\eta^* \mathcal{N}_0^{\#p,q} \simeq \mathcal{N}_0'^{p,q} \supset \psi^* \mathcal{N}_0{}^{p,q}. \tag{4.8.1}$$

On $Y^\#$ we are in the situation where the variation is maximal, hence i) holds true and the dual of the kernel $\mathcal{N}^{\#p,q}_0$ is big. So for any ample invertible sheaf $\mathcal{H}$ we find some $\alpha > 0$ and a morphism

$$\bigoplus^r \mathcal{H} \longrightarrow S^\alpha((\mathcal{N}^{\#p,q}_0)^\vee)$$

which is surjective over some open set. Obviously the same holds true for any invertible sheaf $\mathcal{H}$, independent of the ampleness. In particular we may choose for any $\nu > 1$ with $f_*\omega^\nu_{X/Y} \neq 0$ and for the number N_ν given by 2.8, g) the sheaf

$$\mathcal{H} = \det(g^\#_* \omega^\nu_{Z^\#/Y^\#})^{N_\nu}.$$

By 2.8, f) and by (4.8.1), applied to $Y' \to Y^\#$, the sheaf

$$\begin{aligned}\eta^*(S^\alpha((\mathcal{N}^{\#p,q}_0)^\vee) \otimes \det(g^\#_* \omega^\nu_{Z^\#/Y^\#})^{-N_\nu}) = S^\alpha((\mathcal{N}'^{p,q}_0)^\vee) \otimes \det(g_* \omega^\nu_{Z/Y'})^{-N_\nu} \\ \subset \psi^*(S^\alpha((\mathcal{N}^{p,q}_0)^\vee) \otimes \lambda_\nu^{-1})\end{aligned}$$

is generically generated. By 1.3 the same holds true for some power of

$$S^\alpha((\mathcal{N}^{p,q}_0)^\vee) \otimes \lambda_\nu^{-1}.$$

By 3.3 and by the choice of λ_ν in 2.8, g), one finds $\kappa(\lambda_\nu) \geq \mathrm{Var}(f)$ (see 2.2). $\square$

To finish the proof of 1.4, i) and ii) we just have to repeat the arguments used to prove 4.5, using 4.6. By 4.8 $\mathcal{O}_Y = F_0^{n,0}$ can not lie in the kernel of $\tau_{n,0}$. We choose $1 \leq m \leq n$ to be the largest number with $\tau^m(F_0^{n,0}) \neq 0$. Then $\tau^m(F_0^{n,0})$ is contained in $\mathcal{N}_0^{n-m,m} \otimes S^m(\Omega^1_Y(\log S))$, and we obtain morphisms of sheaves

$$(\mathcal{N}_0^{n-m,m})^\vee \longrightarrow F_0^{n,0} \otimes (\mathcal{N}_0^{n-m,m})^\vee \xrightarrow{\neq 0} S^m(\Omega^1_Y(\log S)). \tag{4.8.2}$$

Under the assumptions made in 1.4, ii) we take $\mathcal{P}$ to be the image of this morphism. By 4.8, i), this is the image of a big sheaf, hence big.

If $\mathrm{Var}(f) < \dim(Y)$ 4.8, ii) implies that $S^\alpha((\mathcal{N}_0^{n-m,m})^\vee) \otimes \lambda^{-1}$ is globally generated, for some $\alpha > 0$, and by (4.8.2) one obtains a non-trivial morphism

$$\bigoplus^r \lambda \longrightarrow S^{\alpha \cdot m}(\Omega^1_Y(\log S)).$$

$\square$

5. Base spaces of families of smooth minimal models

As promised in section one, we will show that in problem 1.5, the bigness in b) follows from the weak positivity in a). The corresponding result holds true for base spaces of morphisms of maximal variation whose fibres are smooth minimal models.

Throughout this section Y denotes a projective manifold, and S a reduced normal crossing divisor in Y.

Corollary 5.1. *Let $f : V \to U = Y \setminus S$ be a smooth family of n-dimensional projective manifolds with* $\mathrm{Var}(f) = \dim(Y)$ *and with $\omega_{V/U}$ f-semi-ample. If $\Omega^1_Y(\log S)$ is weakly positive, then $\omega_Y(S)$ is big.*

Proof. By 1.4, iii), there exists some $m > 0$, a big coherent subsheaf $\mathcal{P}$, and an injective map

$$\mathcal{P} \xrightarrow{\subset} S^m(\Omega^1_Y(\log S)).$$

Its cokernel $\mathcal{C}$, as the quotient of a weakly positive sheaf, is weakly positive, hence $\det(S^m(\Omega^1_Y(\log S)))$ is the tensor product of the big sheaf $\det(\mathcal{P})$ with the weakly positive sheaf $\det(\mathcal{C})$. □

Corollary 5.2 (Kovács, [14], for $S = \emptyset$)**.**
If $T_Y(-\log S)$ is weakly positive, then there exists

a) *no non-isotrivial smooth projective family $f : V \to U$ of canonically polarized manifolds.*
b) *no smooth projective family $f : V \to U$ with* $\mathrm{Var}(f) = \dim(U)$ *and with $\omega_{V/U}$ f-semi-ample.*

Proof. In both cases 1.4 would imply for some $m > 0$ that $S^m(\Omega^1_Y(\log S))$ has a subsheaf $\mathcal{A}$ of positive Kodaira dimension. But $\mathcal{A}^\vee$, as a quotient of a weakly positive sheaf, must be weakly positive, contradicting $\kappa(\mathcal{A}) > 0$. □

There are other examples of varieties U for which $S^m(\Omega^1_Y(\log S))$ can not contain a subsheaf of strictly positive Kodaira dimension or more general, for which

$$H^0(Y, S^m(\Omega^1_Y(\log S))) = 0 \quad \text{for all} \quad m > 0. \tag{5.2.1}$$

The argument used in 5.2 carries over and excludes the existence of families, as in 5.2, a) or b). For example, (5.2.1) has been verified by Brückmann for $U = H$ a complete intersection in $\mathbb{P}^N$ of codimension $\ell < \frac{N}{2}$ (see [3] for example). As a second application of this result, one can exclude certain discriminant loci for families of canonically polarized manifolds in $\mathbb{P}^N$. If $H = H_1 + \cdots + H_\ell$ is a normal crossing divisor in $\mathbb{P}^N$, and $\ell < \frac{N}{2}$, then for $U = \mathbb{P}^N \setminus H$ the conclusions in 5.2 hold true. In order allow a proof by induction, we formulate both results in a slightly more general setup.

Corollary 5.3. *For $\ell < \frac{N}{2}$ let $H = H_1 + \cdots + H_\ell$ be a normal crossing divisor in $\mathbb{P}^N$. For $0 \leq r \leq l$ define*

$$H = \bigcap_{j=r+1}^{\ell} H_j, \quad S_i = H_i|_H, \quad S = \sum_{i=1}^{r} S_i, \quad \textit{and} \quad U = H \setminus S$$

(where for $l = r$ the intersection with empty index set is $H = \mathbb{P}^N$). Then there exists

a) *no non-isotrivial smooth projective family $f : V \to U$ of canonically polarized manifolds.*
b) *no smooth projective family $f : V \to U$ with* $\mathrm{Var}(f) = \dim(U)$ *and with $\omega_{V/U}$ f-semi-ample.*

Proof. Let $\mathcal{A}$ be an invertible sheaf of Kodaira dimension $\kappa(\mathcal{A}) > 0$. Replacing $\mathcal{A}$ by some power, we may assume that $\dim(H^0(H, \mathcal{A})) > r + 1$. We have to verify that there is no injection $\mathcal{A} \to S^m(\Omega^1_H(\log S))$.

For $r = 0$ such an injection would contradict the vanishing (5.2.1) shown in [3]. Hence starting with $r = 0$ we will show the non-existence of the subsheaf $\mathcal{A}$ by induction on $\dim(H) = N - \ell + r$ and on r.

The exact sequence

$$0 \to \Omega^1_{S_r}(\log(S_1 + \cdots + S_{r-1})) \to \Omega^1_H(\log S)|_{S_r} \to \mathcal{O}_{S_r} \to 0$$

induces a filtration on $S^m(\Omega^1_H(\log S))|_{S_r}$ with subsequent quotients

$$S^\mu(\Omega^1_{S_r}(\log(S_1 + \cdots + S_{r-1})))$$

for $\mu = 0, \cdots, m$. By induction none of those quotients can contain an invertible subsheaf of positive Kodaira dimension. Hence either the restriction of $\mathcal{A}$ to S_r is a sheaf with $\kappa(\mathcal{A}|_{S_r}) \leq 0$, hence $\dim(H, \mathcal{A}(-S_r)) > r$, or the image of $\mathcal{A}$ in $S^m(\Omega^1_H(\log S))|_{S_r}$ is zero. In both cases

$$S^m(\Omega^1_H(\log(S)) \otimes \mathcal{O}_H(-S_r)$$

contains an invertible subsheaf $\mathcal{A}_1$ with at least two linearly independent sections, hence of positive Kodaira dimension. Now we repeat the same argument a second time:

$$(S^m(\Omega^1_H(\log S)) \otimes \mathcal{O}_H(-S_r))|_{S_r}$$

has a filtration with subsequent quotients

$$(S^\mu(\Omega^1_{S_r}(\log(S_1 + \cdots + S_{r-1})) \otimes \mathcal{O}_H(-S_r))|_{S_r}).$$

$\mathcal{O}_H(S_r)$ is ample, hence by induction none of those quotients can have a non-trivial section. Repeating this argument m times, we find an invertible sheaf contained in

$$\begin{aligned} S^m(\Omega^1_H(\log S)) \otimes \mathcal{O}_H(-m \cdot S_r) &= S^m(\Omega^1_H(\log S) \otimes \mathcal{O}_H(-S_r)) \\ &\subset S^m(\Omega^1_H(\log(S_1 + \cdots + S_{r-1}))), \end{aligned}$$

and of positive Kodaira dimension, contradicting the induction hypothesis. $\square$

6. Subschemes of moduli stacks of canonically polarized manifolds

Let M_h denote the moduli scheme of canonically polarized n-dimensional manifolds with Hilbert polynomial h. In this section we want to apply 1.4 to obtain properties of submanifolds of the moduli stack. Most of those remain true for base spaces of smooth families with a relatively semi-ample dualizing sheaf, and of maximal variation.

Assumptions 6.1. Let Y be a projective manifold, S a normal crossing divisor and $U = Y - S$. Consider the following three setups:

a) There exists a quasi-finite morphism $\varphi : U \to M_h$ which is induced by a smooth family $f : V \to U$ of canonically polarized manifolds.
b) There exists a smooth family $f : V \to U$ with $\omega_{V/U}$ f-semi-ample and with $\mathrm{Var}(f) = \dim(U)$.
c) There exists a smooth family $f : V \to U$ with $\omega_{V/U}$ f-semi-ample and some $\nu \geq 2$ for which the following holds true. Given a non-singular projective manifold Y', a normal crossing divisors S' in Y', and a quasi-finite morphism $\psi' : U' = Y' \setminus S' \longrightarrow U$, let X' be a non-singular projective compactification of $V \times_U U'$ such that the second projection induces a morphism $f' : X' \to Y'$. Then the sheaf $\det(f'_* \omega^{\nu}_{X'/Y'})$ is ample with respect to U'.

Although we are mainly interested in the cases 6.1, a) and b), we included the quite technical condition c), since this is what we really need in the proofs.

The assumption made in a) implies the one in c). In fact, if φ is quasi-finite, the same holds true for $\varphi \circ \psi' : U' \to M_h$, and by 3.4, iii), $\det(f'_* \omega^{\nu}_{X'/Y'})$ is ample with respect to U'.

Under the assumption b), it might happen that we have to replace U in c) by some smaller open subset $\tilde{U}$. To this aim start with the open set U_g considered in 2.7. Applying 3.4, i), to the family $g : Z \to Y'$ in (2.6.1), one finds an open subset $\tilde{U}'$ of Y' with $g_* \omega^{\nu}_{Z/Y'}$ ample with respect to $\tilde{U}'$. We may assume, of course, that $\tilde{U}'$ is the preimage of $\tilde{U} \subset U_g$. Since $g : Z \to Y'$ is birational to a mild morphism over Y', the same holds true for all larger coverings, and the condition c) follows by flat base change, for $\tilde{U}$ instead of U.

Let us start with a finiteness result for morphisms from curves to M_h, close in spirit to the one obtained in [2], 4.3, in case that M_h is the moduli space of surfaces of general type. Let C be a projective non-singular curve and let C_0 be a dense open subset of C. By [8] the morphisms

$$\pi : C \to Y \quad \text{with} \quad \pi(C_0) \subset U$$

are parameterized by a scheme $\mathbf{H} := \mathbf{Hom}((C, C_0), (Y, U))$, locally of finite type.

Theorem 6.2.

i) *Under the assumptions made in 6.1, a) or c), the scheme* $\mathbf{H}$ *is of finite type.*
ii) *Under the assumption 6.1, b), there exists an open subscheme* U_g *in* U *such that there are only finitely many irreducible components of* $\mathbf{H}$ *which parameterize morphisms* $\pi : C \to Y$ *with* $\pi(C_0) \subset U$ *and* $\pi(C_0) \cap U_g \neq \emptyset$.

Proof. Let us return to the notations introduced in 2.6 and 2.7. There we considered an open dense non-singular subvariety U_g of U, depending on the construction of the diagram (2.6.1). In particular U_g embeds to Y.

Let $\mathcal{H}$ be an ample invertible sheaf on Y. In order to prove i) we have to find some constant c which is an upper bound for $\deg(\pi^* \mathcal{H})$, for all morphisms $\pi : C \to Y$ with $\pi(C_0) \subset U$. For part ii) we have to show the same, under the additional assumption that $\pi(C) \cup U_g \neq \emptyset$. Let us start with the latter.

By 3.4, iii) in case 6.1, a), or by assumption in case 6.1, c), one finds the sheaf λ_ν, defined in 2.6, d), to be ample with respect to U_g. For part ii) of 6.2, i.e. if one just assumes that $\mathrm{Var}(f) = \dim(U)$, we may use 3.3, and choose U_g a bit smaller to guarantee the ampleness of $g_*\omega^\nu_{Z/Y'}$ over $\psi^{-1}(U_g)$.

Replacing N_ν by some multiple and λ_ν by some tensor power, we may assume that $\lambda_\nu \otimes \mathcal{H}^{-1}$ is generated by global sections over U_g.

Assume first that $\pi(C_0) \cap U_g \neq \emptyset$. Let $h : W \to C$ be a morphism between projective manifolds, obtained as a compactification of $X \times_Y C_0 \to C_0$. By definition h is smooth over C_0. In 2.7 we have shown, that

$$\deg(\pi^*\lambda_\nu) \leq N_\nu \cdot \deg(\det(h_*\omega^\nu_{W/C})).$$

On the other hand, upper bounds for the right hand side have been obtained for case a) in [2], [13] and in general in [24]. Using the notations from [24],

$$\deg(\det(h_*\omega^\nu_{W/C})) \leq (n \cdot (2g(C) - 2 + s) + s) \cdot \nu \cdot \mathrm{rank}(h_*\omega^\nu_{W/C}) \cdot e,$$

where $g(C)$ is the genus of C, where $s = \#(C - C_0)$, and where e is a positive constant, depending on the general fibre of h. In fact, if F is a general fibre of h, the constant e can be chosen to be $e(\omega^\nu_F)$. Since the latter is upper semicontinous in smooth families (see [6] or [23], 5.17) there exists some e which works for all possible curves. Altogether, we found an upper bound for $\deg(\pi^*\mathcal{H})$, whenever the image $\pi(C)$ meets the dense open subset U_g of U.

In i), the assumptions made in 6.1, a) and c) are compatible with restriction to subvarieties of U, and we may assume by induction, that we already obtained similar bounds for all curves C with $\pi(C_0) \subset (U \setminus U_g)$. $\square$

From now on we fix again a projective non-singular compactification with $S = Y \setminus U$ a normal crossing divisor. Even if $\varphi : U \to M_h$ is quasi finite, one can not expect $\Omega^1_Y(\log S)$ to be ample with respect to U, except for $n = 1$, i.e. for moduli of curves. For $n > 1$ there are obvious counter examples.

Example 6.3. Let $g_1 : Z_1 \to C_1$ and $g_2 : Z_2 \to C_2$ be two non-isotrivial families of curves over curves C_1 and C_2, with degeneration loci S_1 and S_2, respectively. We assume both families to be semi-stable, of different genus, and we consider the product

$$f : X = Z_1 \times Z_2 \longrightarrow C_1 \times C_2 = Y,$$

the projections $p_i : Y \to C_i$, and the discriminant locus $S = p_1^{-1}(S_1) \cup p_2^{-1}(S_2)$. For two invertible sheaves $\mathcal{L}_i$ on C_i we write

$$\mathcal{L}_1 \boxplus \mathcal{L}_2 = p_1^*\mathcal{L}_1 \oplus p_2^*\mathcal{L}_2 \quad \text{and} \quad \mathcal{L}_1 \boxtimes \mathcal{L}_2 = p_1^*\mathcal{L}_1 \otimes p_2^*\mathcal{L}_2.$$

For example

$$S^2(\mathcal{L}_1 \boxplus \mathcal{L}_2) = p_1^*\mathcal{L}_1^2 \oplus p_2^*\mathcal{L}_2^2 \oplus \mathcal{L}_1 \boxtimes \mathcal{L}_2.$$

The family f is non-isotrivial, and it induces a generically finite morphism to the moduli space of surfaces of general type M_h, for some h. Obviously,

$$\Omega^1_Y(\log S) = \Omega^1_{C_1}(\log S_1) \boxplus \Omega^1_{C_2}(\log S_2) := p_1^*(\Omega^1_{C_1}(\log S_1)) \oplus p_2^*(\Omega^1_{C_2}(\log S_2))$$

can not be ample with respect to any open dense subset.

Let us look, how the edge morphisms $\tau_{p,q}$ defined in section 4 look like in this special case. To avoid conflicting notations, we write $G_i^{p,q}$ instead of $F^{p,q}$, for the two families of curves, and

$$\sigma_i : {G_i}^{1,0} = g_{i*}\mathcal{O}_{Z_i} = \mathcal{O}_{C_i} \longrightarrow {G_i}^{0,1} \otimes \Omega^1_{C_i}(\log S_i)$$

for the edge morphisms. The morphism

$$\tau^2 = \tau_{1,1} \circ \tau_{2,0} : F^{2,0} = \mathcal{O}_Y \longrightarrow F^{0,2} \otimes S^2(\Omega^1_Y(\log S)),$$

considered in the proof of 1.4, i) and ii), thereby induces three maps,

$$t_i : F^{0,2\vee} \to S^2(p_i^*\Omega^1_{C_i}(\log S_i)),$$

for $i = 1,\ 2$, and

$$t : F^{0,2\vee} \longrightarrow \Omega^1_{C_1}(\log S_1) \boxtimes \Omega^1_{C_2}(\log S_2).$$

Since $F^{0,2\vee} = g_{1*}\omega^2_{Z_1/C_1} \boxtimes g_{2*}\omega^2_{Z_2/C_2}$ is ample the first two morphisms t_1 and t_2 must be zero.

$$F^{1,1} = R^1 f_*((\omega_{Z_1/C_1} \boxplus \omega_{Z_2/C_2}) \otimes \omega^{-1}_{X/Y}) \simeq R^1 f_*(\omega^{-1}_{Z_1/C_1} \boxplus \omega^{-1}_{Z_2/C_2}),$$

where the isomorphism interchanges the two factors. In particular

$$F^{1,1} = G_1^{0,1} \boxplus G_2^{0,1},$$

and one has $\tau_{2,0} = \sigma_1 \boxplus \sigma_2$. Its image lies in the direct factor

$$G' := G_1^{0,1} \otimes \Omega^1_{C_1}(\log S_1) \boxplus G_2^{0,1} \otimes \Omega^1_{C_2}(\log S_2)$$

of $F^{1,1} \otimes \Omega^1_Y(\log S)$. The picture should be the following one:

$$F^{0,2} \otimes \Omega^1_Y(\log S) = (G_1^{0,1} \boxtimes G_2^{0,1}) \otimes (\Omega^1_{C_1}(\log S_1) \boxplus \Omega^1_{C_2}(\log S_2)),$$

and $\tau_{1,1}|_{G_1^{0,1}} = \mathrm{id}_{G_1^{0,1}} \otimes \sigma_2$ with image in

$$(G_1^{0,1} \boxtimes G_2^{0,1}) \otimes \Omega^1_{C_2}(\log S_2).$$

Hence $\tau_{1,1} \circ \tau_{2,0}$ is the sum of the two maps $\tau_{1,1}|_{G_i^{0,1}} \circ \sigma_i$, both with image in

$$(G_1^{0,1} \boxtimes G_2^{0,1}) \boxtimes \Omega^1_{C_1}(\log S_1) \otimes \Omega^1_{C_2}(\log S_2).$$

In general, when there exists a generically finite morphism $\varphi : C_1 \times C_2 \to M_h$ induced by $f : V \to U$, the picture should be quite similar, however we were unable to translate this back to properties of the general fibre of f. However, for moduli of surfaces there can not exist a generically finite morphism from the product of three curves. More generally one obtains from 1.4:

Corollary 6.4. *Let $U = C_1^0 \times \cdots C_\ell^0$ be the product of ℓ quasi-projective curves, and assume there exists a smooth family $f : V \to U$ with $\omega_{V/U}$ f-semi-ample and with* $\mathrm{Var}(f) = \dim(U)$. *Then $\ell \leq n = \dim(V) - \dim(U)$.*

Proof. For C_i, the non-singular compactification of C_i^0, and for $S_i = C_i \setminus C_i^0$, a compactification of U is given by $Y = C_1 \times \cdots \times C_\ell$ with boundary divisor $S = \sum_{i=1}^{\ell} pr_i^* S_i$. Then

$$S^m(\Omega^1_Y(\log S)) = \bigoplus S^{j_1}(pr_1^*\Omega^1_{C_1}(\log S_1)) \otimes \cdots \otimes S^{j_\ell}(pr_\ell^*\Omega^1_{C_\ell}(\log S_\ell))$$

where the sum is taken over all tuples $j_1, \ldots, j_\ell$ with $j_1 + \cdots + j_\ell = m$. If $\ell > m$, each of the factors is the pullback of some sheaf on a strictly lower

dimensional product of curves, hence for $\ell > m$ any morphism from a big sheaf $\mathcal{P}$ to $S^m(\Omega^1_Y(\log S))$ must be trivial. If $\psi : Y' \to Y$ is a finite covering, the same holds true for $\psi^* S^m(\Omega^1_Y(\log S))$. By 1.4, iv), there exists such a covering, some $m \leq n$ and a big subsheaf of $\psi^* S^m(\Omega^1_Y(\log S))$, hence $\ell \leq m \leq n$. □

The next application of 1.4 is the rigidity of generic curves in moduli stacks. If $\varphi : U \to M_h$ is induced by a family, 1.4, ii) provides us with a big subsheaf $\mathcal{P}$ of $S^m(\Omega^1_Y(\log S))$, and if we do not insist that $m \leq n$, the same holds true whenever there exists a family $V \to U$, as in 6.1, b). In both cases, replacing m by some multiple, we find an ample invertible sheaf $\mathcal{H}$ on Y and an injection

$$\iota : \bigoplus \mathcal{H} \longrightarrow S^m(\Omega^1_Y(\log S)).$$

Let U_1 be an open dense subset in U, on which ι defines a subbundle.

Corollary 6.5. *Under the assumption a) or b) in 6.1 there exists an open dense subset U_1 and for each point $y \in U_1$ a curve $C_0 \subset U$, passing through y, which is rigid, i.e.: If for a reduced curve T_0 and for $t \in T_0$ there exists a morphism $\rho : T_0 \times C_0 \to U$ with $\rho(\{t_0\} \times C_0) = C_0$, then ρ factors through $pr_2 : T_0 \times C_0 \to C_0$.*

Proof. Let $\pi : \mathbb{P} = \mathbb{P}(\Omega^1_Y(\log S)) \to Y$ be the projective bundle.

$$\iota : \bigoplus \mathcal{H} \longrightarrow \pi_* \mathcal{O}_{\mathbb{P}}(m)$$

defines sections of $\mathcal{O}_Y(m) \otimes \pi^* \mathcal{H}^{-1}$, which are not all identically zero on $\pi^{-1}(y)$ for $y \in U_1$. Hence there exists a non-singular curve $C_0 \subset U$ passing through y, such that the composite

$$\bigoplus \mathcal{H}|_U \longrightarrow S^m(\Omega^1_U) \longrightarrow S^m(\Omega^1_{C_0}) \tag{6.5.1}$$

is surjective over a neighborhood of y. For a nonsingular curve T_0 and $t_0 \in T_0$ consider a morphism $\phi_0 : T_0 \times C_0 \to U$, with $\phi_0(\{t_0\} \times C_0) = C_0$. Let T and C be projective non-singular curves, containing T_0 and C_0 as the complement of divisors Θ and Γ, respectively. On the complement W of a codimension two subset of $T \times C$ the morphism ϕ_0 extends to $\phi : W \to Y$. Then ι induces a morphism

$$\phi^* \bigoplus \mathcal{H} \longrightarrow \phi^* S^m(\Omega^1_Y(\log S)) \longrightarrow S^m(\Omega^1_T(\log \Theta) \boxplus \Omega^1_C(\log \Gamma))|_W$$

whose composite with

$$\begin{aligned} S^m(\Omega^1_T(\log \Theta) \boxplus \Omega^1_C(\log \Gamma))|_W &\longrightarrow S^m(pr_2^* \Omega^1_C(\log \Gamma))|_W \\ &\longrightarrow S^m(\Omega_{\{t\} \times C}(\log \Gamma))|_{W \cap \{t\} \times C} \end{aligned}$$

is non-zero for all t in an open neighborhood of t_0, hence

$$\phi^* \bigoplus \mathcal{H} \longrightarrow pr_2^* S^m(\Omega^1_C(\log \Gamma))|_W$$

is surjective over some open dense subset. Since $\mathcal{H}$ is ample, this is only possible if $\phi : W \to Y$ factors through the second projection $W \to T \times C \to C$. □

Assume we know in 6.5 that $\Omega^1_Y(\log S)$ is ample over some dense open subscheme U_2. Then the morphism (6.5.1) is non-trivial for all curves C_0 meeting U_2, hence the argument used in the proof of 6.5 implies, that a morphisms $\pi : C_0 \to U$ with $\pi(C_0) \cap U_2 \neq \emptyset$ has to be rigid. If $U_2 = U$, this, together with 6.2 proves the next corollary.

Corollary 6.6. *Assume in 6.2, i), that $\Omega^1_Y(\log S)$ is ample with respect to U. Then $\mathbf{H}$ is a finite set of points.*

The generic rigidity in 6.5, together with the finiteness result in 6.2, implies that subvarieties of the moduli stacks have a finite group of automorphism. Again, a similar statement holds true under the assumption 6.1, b).

Theorem 6.7. *Under the assumption 6.1, a) or b) the automorphism group $\mathrm{Aut}(U)$ of U is finite.*

Proof. Assume $\mathrm{Aut}(U)$ is infinite, and choose an infinite countable subgroup

$$G \subset \mathrm{Aut}(U).$$

Let U_1 be the open subset of U considered in 6.5, and let U_g be the open subset from 6.2, b). We may assume that $U_1 \subset U_g$ and write $\Gamma = U \setminus U_1$. Since

$$\bigcup_{g \in G} g(\Gamma) \neq U$$

we can find a point $y \in U_1$ whose G-orbit is an infinite set contained in U_1. By 6.5 there are rigid smooth curves $C_0 \subset U$ passing through y. Obviously, for all $g \in G$ the curve $g(C_0) \subset U$ is again rigid, it meets U_1, hence U_g and the set of those curves is infinite, contradicting 6.2, b). □

7. A vanishing theorem for sections of symmetric powers of logarithmic one forms

Proposition 7.1. *Let Y be a projective manifold and let $D = D_1 + \ldots + D_r$ and $S = S_1 + \ldots + S_\ell$ be two reduced divisors without common component. Assume that*

i) *$S + D$ is a normal crossing divisor.*
ii) *For no subset $J \subseteq \{1, \ldots, \ell\}$ the intersection*

$$S_J = \bigcap_{j \in J} S_i$$

is zero dimensional.

iii) *$T_Y(-\log\, D) = (\Omega^1_Y(\log\, D))^\vee$ is weakly positive over $U_1 = Y - D$.*

Then for all ample invertible sheaves $\mathcal{A}$ and for all $m \geq 1$

$$H^0(Y, S^m(\Omega^1_Y(\log(D+S))) \otimes \mathcal{A}^{-1}) = 0.$$

Corollary 7.2. *Under the assumption i), ii) and iii) in 7.1 there exists no smooth family $f : V \to U = Y \setminus (S + D)$ with $\mathrm{Var}(f) = \dim Y$. In particular there is no generically finite morphism $U \to M_h$, induced by a family.*

Proof. By 1.4, iii) the existence of such a family implies that for some $m > 0$ the sheaf $S^m(\Omega^1_Y(\log(D+S)))$ contains a big coherent subsheaf $\mathcal{P}$. Replacing m by some multiple, one can assume that $\mathcal{P}$ is ample and invertible, contradicting 7.1. □

Proof of 0.2 and of the second part of 0.4. Since for an abelian variety Y the sheaf Ω^1_Y is trivial, and since the condition ii) in 7.1 is obvious for $\ell < \dim Y$, part a) of 0.2 is a special case of 7.2.

For b) again i) and ii) hold true by assumption. For iii) we remark, that

$$\Omega^1_{\mathbb{P}^{\nu_i}}(\log\, D^{(\nu_i)}) = \oplus^{\nu_i}\mathcal{O}_{\mathbb{P}^{\nu_i}},$$

hence $\Omega^1_Y(\log\, D)$ is again a direct sum of copies of $\mathcal{O}_Y$. Assume that $\mathcal{A}$ is an invertible subsheaf of $S^m(\Omega^1_Y(\log(D+S)))$, for some $m > 0$. If $\kappa(\mathcal{A}) > 0$, then for some $\mu_i \in \mathbb{N}$,

$$\mathcal{A} = \mathcal{O}_Y(\mu_1, \ldots, \mu_k) = \bigotimes_{i=1}^{k} pr_i^* \mathcal{O}_{\mathbb{P}^{\nu_i}}(\mu_i).$$

By 7.1 not all the μ_i can be strictly larger than zero, hence

$$\kappa(\mathcal{A}) \leq \mathrm{Max}\{\dim(Y) - \nu_i;\ i = 1, \ldots, k\} = M.$$

By 1.4, i), for any morphism $\varphi : U \to M_h$, induced by a family $f : V \to U$, one has

$$\mathrm{Var}(f) = \dim(\varphi(U)) \leq M.$$

1.4, iii), implies that there exists no smooth family $f : V \to U$ of maximal variation, and with $\omega_{V/U}$ f-semi-ample. □

Remark 7.3. In 0.2, a), one can also show, that for an abelian variety Y and for a morphism $\varphi : Y \to M_h$, induced by a family,

$$\dim(\varphi(Y)) \leq \dim(Y) - \nu,$$

where ν is the dimension of the smallest simple abelian subvariety of Y. In fact, by the Poincaré decomposition theorem, Y is isogenous to the product of simple abelian varieties, hence replacing Y by an étale covering, we may assume that

$$Y = Y_1 \times \ldots \times Y_k$$

with Y_i simple abelian, and with

$$\nu = \dim(Y_1) \leq \dim(Y_2) \leq \ldots \leq \dim Y_k.$$

Since an invertible sheaf of positive Kodaira dimension on a simple abelian variety must be ample, one finds that for an non-ample invertible sheaf $\mathcal{A}$ on Y

$$\kappa(\mathcal{A}) \leq \dim(Y) - \nu.$$

Before proving 7.1 let us show that for $Y = \mathbb{P}^2$ we can not allow S to have two irreducible components of high degree, even if $D = 0$.

Example 7.4. Given a surface Y and a normal crossing divisor $S+D$, with $S=\sum_{i=1}^{\ell} S_i$, consider the two exact sequences

$$0 \to \mathcal{O}_Y(-S_i) \to \mathcal{O}_Y \to \mathcal{O}_{S_i} \to 0 \qquad \text{and}$$

$$0 \to \Omega^1_Y(\log D) \to \Omega^1_Y(\log(D+S)) \to \bigoplus_{i=1}^{\ell} \mathcal{O}_{S_i} \to 0.$$

Writing $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E})$ for a sheaf $\mathcal{E}$ on Y, one finds

$$c(\mathcal{O}_{S_i}) = 1 + S_i + S_i^2$$

and

$$c(\Omega^1_Y(\log(D+S))) = c(\Omega^1_Y(\log D)) \cdot \prod_{i=1}^{\ell}(1 + S_i + S_i^2).$$

Hence

$$c_2(\Omega^1_Y(\log(D+S))) = c_2(\Omega^1_Y(\log D)) + \sum_{i=1}^{\ell} c_1(\Omega^1_Y(\log D)).S_i + \sum_{i<j} S_i.S_j + \sum_{i=1}^{\ell} S_i^2$$

and $c_1(\Omega^1_Y(\log(D+S))) = c_1(\Omega^1_Y) + D + S$. The Riemann-Roch theorem for vector bundles on surfaces and the isomorphism

$$S^m(\Omega^1_Y(\log(D+S))^\vee) \otimes \omega_Y = S^m(\Omega^1_Y(\log(D+S))) \otimes \omega_Y \otimes \omega_Y(D+S)^{-m}$$

imply that for an invertible sheaf $\mathcal{A}$

$$\begin{aligned} &h^0(Y, S^m(\Omega^1_Y(\log(D+S))) \otimes \mathcal{A}^{-1}) \\ &\qquad + h^0(Y, S^m(\Omega^1_Y(\log(D+S))) \otimes \omega_Y \otimes \omega_Y(D+S)^{-m} \otimes \mathcal{A}) \\ &\qquad\qquad \geq \frac{m^3}{6}(c_1(\Omega^1_Y(\log(D+S)))^2 - c_2(\Omega^1_Y(\log(D+S)))) + O(m^2), \end{aligned}$$

where $O(m^2)$ is a sum of terms of order ≤ 2 in m. If $\omega_Y(D+S)$ is big and if

$$c_1(\Omega^1_Y(\log(D+S)))^2 > c_2(\Omega^1_Y(\log(D+S))) \tag{7.4.1}$$

then for $m \gg 0$ the sheaf $\omega_Y \otimes \omega_Y(D+S)^{-m} \otimes \mathcal{A}$ is a subsheaf of $\mathcal{A}^{-1}$ and

$$h^0(Y, S^m(\Omega^1_Y(\log(D+S))) \otimes \mathcal{A}^{-1}) \neq 0.$$

For $Y = \mathbb{P}^2$ and for a coordinate system $D = D_0 + D_1 + D_2$,

$$\begin{aligned} &c_1(\Omega^1_Y(\log(D+S)))^2 - c_2(\Omega^1_Y(\log D + S))) \\ &\qquad = (\sum_{i=1}^{\ell} S_i)^2 - \sum_{i=1}^{\ell} S_i^2 - \sum_{i<j} S_i.S_j = \sum_{i<j} S_i.S_j, \end{aligned}$$

and as soon as S has more than one component, (7.4.1) holds true. So in 7.1, for $Y = \mathbb{P}^2$ and $\ell > 1$, the arguments used to proof 0.2 fail.

We do not know, whether $U = \mathbb{P}^2 \setminus (S_1 + S_2)$ can be the base of a non-isotrivial family of canonically polarized manifolds.

For $D = 0$ and $Y = \mathbb{P}^2$, one finds

$$\begin{aligned} c_1(\Omega^1_Y(\log S))^2 - c_2(\Omega^1_Y(\log S)) &= 6 - 3 \cdot \deg(S) + \sum_{i<j} S_i.S_j \\ &= 3 \cdot (2 - \deg(S)) + \sum_{i<j} \deg(S_i) \cdot \deg(S_j). \end{aligned}$$

Assume that $2 \leq \deg(S_1) \leq \deg(S_2) \ldots \leq \deg(S_\ell)$. Then the only cases where (7.4.1) does not hold true are $\ell = 2$ and $\deg(S_1) = 2$, or $\ell = 3$ and $\deg(S_i) = 2$ for $i = 1, 2, 3$. Again we do not know any example of a non-isotrivial family over $U = \mathbb{P}^2 \setminus S$.

As a first step in the proof of 7.1 we need

Lemma 7.5. *Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves on Y. Assume that, for a non-singular divisor B, for some ample invertible sheaf $\mathcal{A}$, and for all $m \geq 0$*

$$H^0(Y, S^m(\mathcal{F}) \otimes \mathcal{A}^{-1}) = H^0(B, S^m(\mathcal{F}) \otimes \mathcal{A}^{-1}|_B) = 0.$$

Assume moreover that there exists an exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O}_B \to 0.$$

Then for all $m > 0$

$$H^0(Y, S^m(\mathcal{E}) \otimes \mathcal{A}^{-1}) = 0.$$

Proof. Write $\pi : \mathbb{P} = \mathbb{P}(\mathcal{E}) \to Y$. The surjection $\mathcal{E} \to \mathcal{O}_B$ defines a morphism $s : B \to \mathbb{P}$. For the ideal I of $s(B)$ the induced morphism $\pi^*\mathcal{F} \to I \otimes \mathcal{O}_{\mathbb{P}}(1)$ is surjective, as well as the composite

$$\tilde{\pi}^*\mathcal{F} \to \delta^*(I \otimes \mathcal{O}_{\mathbb{P}}(1)) \to \mathcal{O}_{\tilde{\mathbb{P}}}(-E) \otimes \delta^*\mathcal{O}_{\mathbb{P}}(1),$$

where $\delta : \tilde{\mathbb{P}} \to \mathbb{P}$ is the blowing up of I with exceptional divisor E, and where $\tilde{\pi} = \pi \circ \delta$.

Let us write $M + 1$ for the rank of $\mathcal{E}$. For $y \in B$ and $p = s(y)$ let

$$\delta_y : \tilde{\mathbb{P}}_y \to \mathbb{P}^M = \pi^{-1}(y)$$

be the blowing up of p, with exceptional divisor F. Then

$$\tilde{\pi}^{-1}(y) = \tilde{\mathbb{P}}_y \cup \mathbb{P}^M \quad \text{with} \quad F = \tilde{\mathbb{P}}_y \cap \mathbb{P}^M.$$

In particular $\tilde{\pi}$ is equidimensional, hence flat. For $0 \leq \mu \leq m$ and for $i > 0$

$$H^i(\tilde{\mathbb{P}}_y, \mathcal{O}_{\tilde{\mathbb{P}}_y}(-(\mu + 1) \cdot F) \otimes \delta_y^*\mathcal{O}_{\mathbb{P}^M}(m)) = 0$$

and

$$H^0(\tilde{\mathbb{P}}_y, \mathcal{O}_{\tilde{\mathbb{P}}_y}(-(\mu + 1) \cdot F) \otimes \delta_y^*\mathcal{O}_{\mathbb{P}^M}(\mu)) = 0.$$

One has an exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_{\tilde{\mathbb{P}}_y}&(-(\mu + 1) \cdot F) \otimes \delta_y^*\mathcal{O}_{\mathbb{P}^M}(m) \\ &\to \mathcal{O}_{\tilde{\mathbb{P}}}(-\mu \cdot E) \otimes \delta^*\mathcal{O}_{\mathbb{P}}(m)|_{\tilde{\pi}^{-1}(y)} \to \mathcal{O}_{\mathbb{P}^M}(\mu) \to 0 \end{aligned}$$

and $H^1(\tilde{\pi}^{-1}(y), \mathcal{O}_{\tilde{\mathbb{P}}}(-\mu \cdot E) \otimes \delta^*\mathcal{O}_{\mathbb{P}}(m)|_{\tilde{\pi}^{-1}(y)}) = 0$. By flat base change one finds

$$R^1\tilde{\pi}_*(\mathcal{O}_{\tilde{\mathbb{P}}}(-\mu \cdot E) \otimes \delta^*\mathcal{O}_{\mathbb{P}}(m)) = 0.$$

Moreover
$$\tilde{\pi}_*(\mathcal{O}_{\tilde{\mathbb{P}}}(-\mu\cdot E)\otimes\delta^*\mathcal{O}_{\mathbb{P}}(m))|_y \to \tilde{\pi}_*\mathcal{O}_E(-\mu\cdot E)|_y \cong H^0(\mathbb{P}^M,\mathcal{O}_{\mathbb{P}^M}(\mu))$$
is an isomorphism. The inclusion
$$S^\mu(\mathcal{F}) \longrightarrow \tilde{\pi}_*\delta^*\mathcal{O}_{\mathbb{P}}(\mu)) \cong S^\mu(\mathcal{E})$$
factors through
$$S^\mu(\mathcal{F}) \xrightarrow{\subset} \tilde{\pi}_*(\mathcal{O}_{\tilde{\mathbb{P}}}(-\mu\cdot E)\otimes\delta^*\mathcal{O}_{\mathbb{P}}(\mu)).$$
This map is an isomorphism. We know the surjectivity of
$$\mathcal{F}|_y \longrightarrow \tilde{\pi}_*(\mathcal{O}_{\tilde{\mathbb{P}}}(-E)\otimes\delta^*\mathcal{O}_{\mathbb{P}}(1))|_y \cong H^0(\mathbb{P}^M,\mathcal{O}_{\mathbb{P}^M}(1)),$$
so for $\mu>1$ the morphism from $S^\mu(\mathcal{F})|_y$ to
$$\tilde{\pi}_*(\mathcal{O}_{\tilde{\mathbb{P}}}(-\mu\cdot E)\otimes\delta^*\mathcal{O}_{\mathbb{P}}(\mu))|_y \cong H^0(\mathbb{P}^M,\mathcal{O}_{\mathbb{P}^M}(\mu)) = S^\mu(H^0(\mathbb{P}^M,\mathcal{O}_{\mathbb{P}^M}(1)))$$
is surjective as well. By the choice of $s(B)$ one has
$$\mathcal{O}_{\mathbb{P}}(1)|_{s(B)} = \mathcal{O}_{s(B)} \quad\text{and}\quad \delta^*\mathcal{O}_{\mathbb{P}}(1)|_E = \mathcal{O}_E.$$
Starting with $\mu=m$, assume by descending induction that
$$H^0(Y,\tilde{\pi}_*(\mathcal{O}_{\tilde{\mathbb{P}}}(-\mu\cdot E)\otimes\delta^*\mathcal{O}_{\mathbb{P}}(m))\otimes\mathcal{A}^{-1}) = 0.$$
Since
$$\begin{aligned}H^0(E,\tilde{\pi}_*(\mathcal{O}_{\tilde{\mathbb{P}}}(-(\mu-1)\cdot E)\otimes\delta^*\mathcal{O}_{\mathbb{P}}(m))\otimes\mathcal{A}^{-1}|_E)&\\ = H^0(E,\tilde{\pi}_*(\mathcal{O}_{\tilde{\mathbb{P}}}(-(\mu-1)\cdot E)\otimes\delta^*\mathcal{O}_{\mathbb{P}}(\mu-1))\otimes\mathcal{A}^{-1}|_E)&\\ = H^0(B,S^{\mu-1}(\mathcal{F})\otimes\mathcal{A}^{-1}|_B) = 0&\end{aligned}$$
one finds
$$H^0(Y,\tilde{\pi}_*(\mathcal{O}_{\tilde{\mathbb{P}}}(-(\mu-1)\cdot E)\otimes\delta^*\mathcal{O}_{\mathbb{P}}(m))\otimes\mathcal{A}^{-1} = 0,$$
as well. □

Proof of 7.1. Let us fix some $J\subseteq\{1,\ldots,n\}$ with $S_J\neq\emptyset$. We will write $S_\emptyset = Y$. The sheaf $S^m(T_Y(-\log D))\otimes\mathcal{A}$ is ample with respect to $U_1 = Y-D$. Since $S+D$ is a normal crossing divisor, $S_J\cap U_1\neq\emptyset$ and since $\dim(S_J)\geq 1$,
$$H^0(S_J,S^m(\Omega^1_Y(\log\, D))\otimes\mathcal{A}^{-1}|_{S_J}) = 0.$$
Assume, by induction on ρ, that
$$H^0(S_{J'},S^m(\Omega^1_Y(\log(D+S_1+\ldots+S_{\rho-1})))\otimes\mathcal{A}^{-1}|_{S_{J'}}) = 0,$$
for all $m\geq 0$, and all $J'\subseteq\{\rho,\ldots,\ell\}$ with $S_{J'}\neq\emptyset$. For $J\subseteq\{\rho+1,\ldots,\ell\}$ assume $T=S_J\neq\emptyset$. If $T_\rho = S_{J\cup\{\rho\}}=\emptyset$, i.e. if $S_\rho\cap T=\emptyset$, then
$$\Omega^1_Y(\log(D+S_1+\ldots+S_\rho))|_{S_J} = \Omega^1_Y(\log(D+S_1+\ldots+S_{\rho-1}))|_{S_J}$$
and there is nothing to prove. Otherwise $T_\rho = S_\rho|_T$ is a divisor and the restriction of
$$0\to\Omega^1_Y(\log(D+S_1+\ldots+S_{\rho-1}))\to\Omega^1_Y(\log(D+S_1+\ldots+S_\rho))\to\mathcal{O}_{S_\rho}\to 0$$
to T remains exact. Hence for
$$\mathcal{F} = \Omega^1_Y(\log(D+S_1+\ldots+S_{\rho-1}))|_T \quad\text{and}\quad \mathcal{E} = \Omega^1_Y(\log(D+S_1+\ldots+S_\rho))|_T$$
$$0\to\mathcal{F}\to\mathcal{E}\to\mathcal{O}_{T_S}\to 0$$

is exact, $H^0(T, S^m(\mathcal{F}) \otimes \mathcal{A}^{-1}|_T) = 0$ and

$$H^0(T_\rho, S^m(\mathcal{F}) \otimes \mathcal{A}^{-1}|_{T_S}) = H^0(S_{J\cup\{\rho\}}, S^m(\Omega^1_Y(\log(D + S_1 + \ldots + S_{\rho-1})) \otimes \mathcal{A}^{-1}|_{S_{J\cup\{\rho\}}}) = 0.$$

Using 7.5 we obtain

$$H^0(S_J, S^m(\Omega^1_Y(\log(D + S_1 + \ldots + S_\rho)) \otimes \mathcal{A}^{-1}|_{S_J}) = 0.$$

□

Remark 7.6. The assumption "$\mathcal{A}$ ample" was not really needed in the proof of 7.1. It is sufficient to assume that

$$\kappa(\mathcal{A}|_{S_J}) \geq 1, \text{ for all } J \text{ with } S_J \neq \emptyset.$$

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